ARTICLE 31

The Musical Nature of the Polyhedral Tree of Life

by

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Abstract

The polyhedral Tree of Life is correlated with the eight Church musical modes. These musical scales, which start and finish with the D scale, comprise 48 notes between the tonic and octave — a set of 50 notes that is represented by the 50 corners of the first (6+6) regular polygons of the inner Tree of Life. They comprise 14 different notes (eight with tone ratios of the Pythagorean scale, six with non-Pythagorean tone ratios). The eight scales have 168 rising intervals and 168 falling intervals between notes above the tonic. These intervals are symbolised by the (168+168) yods in the first (6+6) polygons other than their corners. These are the musical and Tree of Life counterparts of the 168 automorphisms and 168 antiautomorphisms of the Klein Quartic. Its symmetry group PSL(2,7) is isomorphic to the symmetry group SL(3,2) of the Fano Plane representing the multiplication of octonions. Their group of automorphisms is the exceptional group G_2. Its seven pairs of roots correlate with the seven pairs of notes in the seven musical scales, explaining why the correspondence exists, viz. Pythagorean music and physics based upon octonions share the same principles. The 48 vertices of the 144 Polyhedron that stem from the 48 faces of the underlying disdyakis dodecahedron express the 48 basic degrees of freedom manifested by a holistic system. In the context of the musical modes, these are notes, which can be grouped into eight sets of six notes between the tonic and octave. The icosahedron with 12 B vertices in the disdyakis triacontahedron represents the 12 different notes between the tonic and octave of these scales. Eight of them consist of fours pairs of notes and their complements, one of which is Pythagorean and the other non-Pythagorean. The 216 edges of the 144 Polyhedron represent the 216 intervals other than octaves between the notes of the eight Church musical modes. The 180 edges of the disdyakis triacontahedron represent the 12 basic notes and the 168 intervals other than octaves between notes above the tonic. The 13 rising intervals and the 13 falling intervals between the tonic and the other types of notes in the Church modes are the musical counterparts of the 13 Catalan solids and their duals. Just as the disdyakis triacontahedron has 26 sheets of vertices that are perpendicular to either B-B or C-C axes and 33 sheets perpendicular to A-A, B-B & C-C axes, so it is the 26th member of the family of Archimedean and Catalan solids and the 33rd stage in the development of polyhedra. The disdyakis triacontahedron contains 1680 vertices, edges & triangles, where 1680 is the number of yods in the lowest 33 Trees of Life constructed from tetractyses. The 28 polyhedra defined by its 62 vertices have 3360 hexagonal yods in their faces. This is the number of yods needed to construct the inner Tree of Life from 2nd-order tetractyses. It is also the number of helical turns in one revolution of the ten whorls of the basic unit of matter described over a century ago by Annie Besant & C.W. Leadbeater, showing how the disdyakis triacontahedron embodies geometrically the structural parameter of this object, identified in earlier work by the author as the $E_8 \times E_8$ heterotic superstring constituent of up and down quarks.

1. The ancient musical scales/Church modes

Historically speaking, musical scales were always divided into eight notes because the ancient Greeks regarded them as composed of two tetrachords (sets of four notes). If the pitch, or 'tone ratio,' of the starting note ('tonic') of a scale is given the value of 1, the eighth note of the scale ('octave') has a tone ratio of 2, that is, it has twice the frequency of the tonic and is the tonic of the next higher set of eight notes. The arithmetic mean of these two frequencies is (1+2)/2 = 3/2. This is the tone ratio of the 'perfect fifth,' so-called because it is the fifth note in the ascending scale, counting from the tonic. The musical scale based entirely on octaves and fifths is called the 'diatonic scale.' The tone ratios of the eight notes making up an octave of this scale are:

$$1 \qquad 9/8 \qquad (9/8)^2 \qquad 4/3 \qquad 3/2 \qquad 27/16 \qquad 243/128 \qquad 2$$
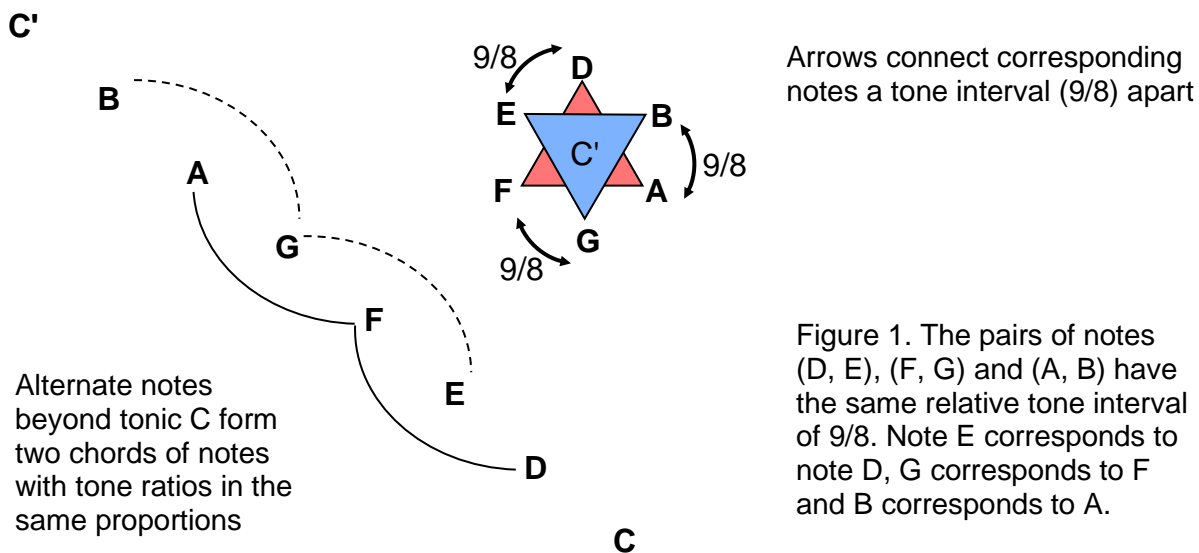
This diatonic scale is also called the 'Pythagorean scale' because Pythagoras is generally thought to have discovered its mathematical basis. It comprises five tone intervals (T) of 9/8 and two intervals (L) of 256/243, called in Greek the *leimma*, or 'left over,' which corresponds to the modern semi-tone, although slightly less than it. The interval pattern of the Pythagorean scale is:

tone–tone–leimma–tone–tone–tone–leimma.
T   T   L   T   T   T   L

In the C scale, the tonic is labelled 'C' and subsequent notes in the scale are labelled D, E, F, G, A & B, the octave being written as C'. Their tone ratios are:

1	9/8	$(9/8)^2$	4/3	3/2	27/16	243/128	2
C	D	E	F	G	A	B	C'

As proved in Article 14,[1] the six notes above the tonic of the C scale can form only two triplets of notes with the same proportions of their tone ratios. They are (D, F, A) and (E, G, B), corresponding members of which are separated by a tone interval (Fig. 1).

Arrows connect corresponding notes a tone interval (9/8) apart

Alternate notes beyond tonic C form two chords of notes with tone ratios in the same proportions

Figure 1. The pairs of notes (D, E), (F, G) and (A, B) have the same relative tone interval of 9/8. Note E corresponds to note D, G corresponds to F and B corresponds to A.

The seven notes D–C' above the tonic of the C scale therefore comprise three pairs of notes (D, E), (F, G) & (A, B) and the octave C' as well as a (3:3:1) pattern. The two triplets (D, F, A) and (E, G, B) correspond in the Tree of Life (Fig. 2) to the two triads of Sephiroth of Construction: Chesed-Geburah-Tiphareth and Netzach-Hod-Yesod, whilst the last note of the C scale, the octave C', corresponds to Malkuth, the last Sephirah of Construction, which completes the emanation of the Tree of Life. The three pairs of notes (D, E), (F, G) & (A, B) separated by a tone interval correspond to the pairs of Sephiroth of Construction on the three pillars of the Tree of Life. This parallelism

suggests that the musical scales, both collectively and in their mathematically perfect version — the Pythagorean scale, conform to the Tree of Life, the Kabbalistic representation of 'Heavenly Man.' This article will add to the evidence presented in
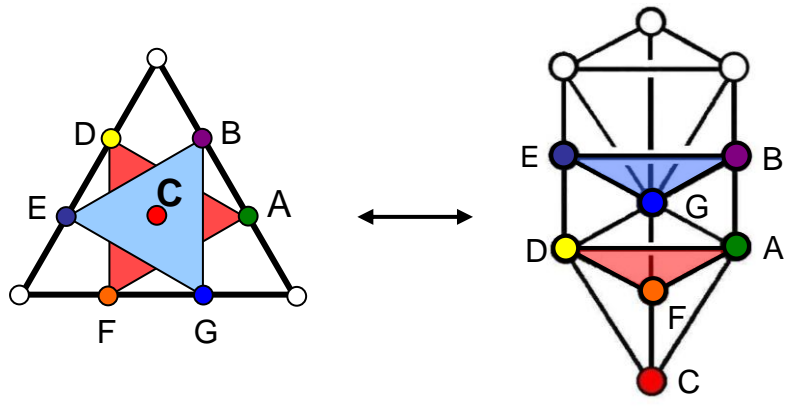

Figure 2. Equivalence between the Tree of Life and the tetractys allows the octave and the three pairs of notes in the Pythagorean scale separated by a tone interval to be assigned to the seven Sephiroth of Construction and to the seven hexagonal yods.

Article 14 for this bold assertion by showing that a remarkable similarity exists between the intervallic composition of the musical scales and the disdyakis triacontahedron, the most complex of the Catalan solids, which was proven in Articles 22-30 to be one of the

Church Musical Modes

Figure 3. The pattern of intervals of each Authentic Mode is the reverse, or mirror image, of a Plagal Mode linked by an arrow.

two polyhedra that make up the polyhedral representation of what the author has called the "inner Tree of Life" (to be discussed later).

Many modern musical scholars hold the view that the ancient Greek musical modes, such as the Dorian and Phrygian modes, were not different octave species but different

keys of the same scale. Article 16 refuted this belief with two arguments:

1. the ancient Greeks would not have used two different musical terms — 'harmonia' and 'tonos' — to mean the same thing, namely, key, instead of scale and key;

2. the Pythagorean mathematician Nichomachus stated[2] (quoting Pythagoras and Plato) that the ancient Egyptians ascribed 28 sounds to the universe, indicating that they were aware of the seven octave species, which have 28 notes with tone ratios belonging to the Pythagorean scale. As it is known[3] that many musicians of ancient
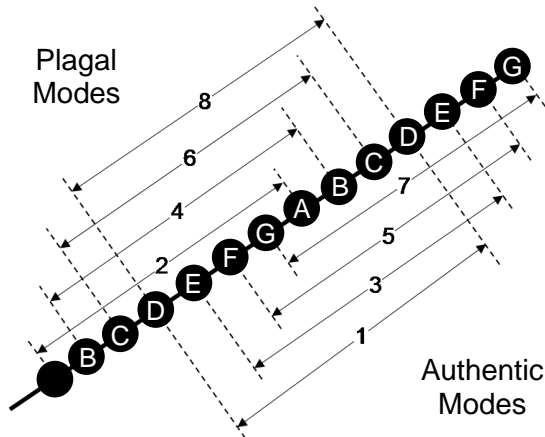

Figure 4. The four Authentic Modes and the four Plagal Modes of Church plainsong.

Greece, as well as learned men like Pythagoras and Plato, studied music with the Egyptians, it is highly unlikely that they neither learnt of the existence of the seven octave species nor played music in various modes based upon them.

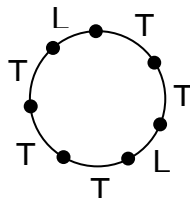The musical modes that have served as the basis of plainsong in the Roman Catholic Church originated not from the ancient Greek modes described by Plato and Aristotle but from the Byzantine octoechoi, which were four pairs of melodic motives that expressed different feelings, or *ethos*. The church modes consist of four 'Authentic Modes' (so-called because they were given the ancient Greek names of Dorian, Phrygian, Lydian and Mixolydian) and four 'Plagal Modes' (Fig. 3). The latter are separated from the former by a perfect fourth (Fig. 4). By dividing a circle into seven arcs (Fig. 5) representing successive intervals of the Pythagorean scale and selecting consecutive sequences of seven intervals as each scale, it is readily seen that there can be only seven different musical scales because there are seven intervals between successive notes in a scale and so the eighth sequence merely repeats the first one:

Figure 5. The circle of five tone intervals (T) and two leimmas (L).

Hypolydian	C' scale			T	T	L	T	T	T	L
Hypophrygian	B scale		L	T	T	L	T	T	T	
Hypodorian	A scale		T	L	T	T	L	T	T	
Mixolydian	G scale		T	T	L	T	T	L	T	
Lydian	F scale	T	T	T	L	T	T	L		
Phrygian	E scale	L	T	T	T	L	T	T		
Dorian	D scale	T	L	T	T	T	L	T		
Hypolydian	C scale	T	T	L	T	T	T	L		

Note:	C	D	E	F	G	A	B	C'	D'	E'	F'	G'	A'	B'	C''
Tone interval:		T	T	L	T	T	T	L	T	T	L	T	T	T	L

Moreover, as all intervals are selected, their sequences do not depend upon the starting

point on the circle. The eight Church modes are labelled 1–8 (see Fig. 4) and start with the D scale. It is called the 'Dorian mode,' although whether it was what the ancient Greeks understand as this mode is unknown. It turns out that the numbering of modes is not arbitrary because, starting with the tonic of the D scale, the last of the different scales is the Pythagorean scale (C scale), the only scale with *all* its notes having tone ratios with Pythagorean values. As it displays the most mathematical harmonies, it is intuitively natural that the last of any sequence of these scales should be the Pythagorean scale. The D scale, however, is the only scale whose pattern of intervals is symmetric. Fig. 3 indicates that it is the same as its mirror image and that Modes 3, 5 & 7 are the mirror image of Modes 6, 4 & 2. This unique property of the D scale gives it a pivotal role in establishing the connection between the musical scales and the polyhedral Tree of Life, as will be explained later. If each rising interval in a scale is replaced by its falling counterpart, i.e., $T \rightarrow T^{-1}$ and $L \rightarrow L^{-1}$, an ascending scale, e.g., TLTTLTT, becomes $T^{-1}L^{-1}T^{-1}T^{-1}L^{-1}T^{-1}T^{-1}$, which is the descending version of the scale TTLTTLT. Inverting each interval of a scale creates another scale whose pattern of intervals is its mirror image. Only the D scale is unchanged when its intervals are inverted.

The eight C–C' scales shown above (or eight consecutive scales starting with any note) span **15** successive notes C–C" and 14 intervals, showing how the Godname YAH with number value **15⁴** prescribes the range of notes needed to define the complete set of four Authentic Modes and four Plagal Modes.

Their (8×7=56) intervals comprise 16 leimmas (L) and 40 whole tones (T), where

$$16 = 4^2 = \begin{array}{c} 4 \quad\quad 4 \\ \boxed{} \\ 4 \quad\quad 4 \end{array}$$

and

$$40 = 4 \times 10 = 4(1+2+3+4) = \begin{array}{c} 4 \quad\quad 8 \\ \boxed{} \\ 16 \quad\quad 12 \end{array}$$

This demonstrates how the Tetrad (4) symbolised by the square expresses the tone interval and leimma composition of the eight scales. Notice that, as each scale has the same number of tone intervals and the same number of leimmas, this composition is the same whatever the scale chosen to start the sequence, e.g., 40 whole tones and 16 leimmas are spanned by the eight scales that start and end with the D-scale.

Table 1. The tone ratios of the eight scales C–C'.

Musical scale	Tone ratio							
C' scale	1	9/8	81/64	4/3	3/2	27/16	243/128	2
B scale	1	256/243	32/27	4/3	1024/729	128/81	16/9	2
A scale	1	9/8	32/27	4/3	3/2	128/81	16/9	2
G scale	1	9/8	81/64	4/3	3/2	27/16	16/9	2
F scale	1	9/8	81/64	729/512	3/2	27/16	243/128	2
E scale	1	256/243	32/27	4/3	3/2	128/81	16/9	2
D scale	1	9/8	32/27	4/3	3/2	27/16	16/9	2
C scale	1	9/8	81/64	4/3	3/2	27/16	243/128	2

(White cells denote Pythagorean notes; red cells denote non-Pythagorean notes).

The eight C–C' musical scales have **48** notes with Pythagorean tone ratios (white cells in Table 1) and 16 notes with non-Pythagorean tone ratios (red cells). Table 2 lists their numbers in each scale. These notes can be represented by an octagon whose sectors
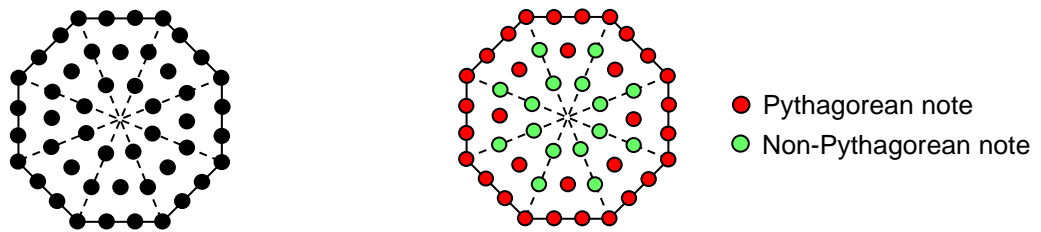
The **48** (●) yods surrounding the centre of an octagon divided into 8 tetractyses symbolize the **48** notes between the tonic and octave in the eight Church musical modes. Each tetractys denotes a mode, the 6 yods per tetractys denoting the 6 notes between the tonic and octave.

The 32 (●) yods symbolize the 32 Pythagorean notes between the tonic and the octave.

The 16 (●) yods symbolize the 16 non-Pythagorean notes between the tonic and the octave.

● Pythagorean note
○ Non-Pythagorean note

Figure 6. The Pythagorean division of the octagon generates the Pythagorean and non-Pythagorean notes of the eight C–C' musical scales.

are tetractyses (Fig. 6) because **48** yods surround its centre. The Pythagorean notes consist of eight tonics (1), eight octaves (2) and 32 notes between the tonic and octave with intermediate tone ratios. They comprise **26** notes of the seven scales C–B and the

Table 2. Number of Pythagorean/non-Pythagorean tone ratios in the eight modes.

Musical scale	Number of Pythagorean tone ratios (white cells)	Number of non-Pythagorean tone ratios (red cells)
C' scale	8	0
B scale	3	5
A scale	5	3
G scale	7	1
F scale	7	1
E scale	4	4
D scale	6	2
C scale	8	0

Total = **48**   Total = 16

six notes of the repeated C' scale. The number of independent (non-repeated) Pythagorean notes is **26**, which is therefore prescribed by the Godname YAHWEH (YHVH), the values of the letters of the Godname denoting the numbers of tones with tone ratios of two possible values:

$$26 = 5(9/8) + 6(3/2) + 3(81/64) + 2(243/128) + 6(4/3) + 4(27/16)$$

5   6   5   10

H   V   H   Y

The letter values of YAHWEH also define the numbers of Pythagorean notes between

the tonic and octave of the seven independent scales:

```
        F scale     5 ——— H
        C scale     6 ——— V
        G scale     5 ——— H
        D scale     4 ⎫
        A scale     3 ⎬ 10   Y
        E scale     2 ⎪
        B scale     1 ⎭
        Total = 26 = YHVH.
```

Including their octaves, there are 33 Pythagorean notes above the tonic of the seven scales. The numbers **26** and 33 measure, respectively, the number of combinations and the number of permutations of 10 objects arranged in a tetractys:

	Number of combinations	**Number of permutations**
a	$2^1 - 1 = 1$	$1! = 1$
b c	$2^2 - 1 = 3$	$2! = 2$
d e f	$2^3 - 1 = 7$	$3! = 6$
g h l j	$2^4 - 1 = 15$	$4! = 24$
	Total = **26**	Total = 33

Indeed, Table 1 indicates that those types of Pythagorean and non-Pythagorean tones above the tonic that occur more than once are *themselves* 10 in number:

```
                        9/8
                81/64       4/3
            3/2     27/16   243/128
    256/243   32/27   128/81   16/9
```

The seven distinct octave species have notes above the tonic with 13 different tone ratios listed below in increasing order of magnitude:

256/243   9/8   32/27   81/64   4/3   1024/729   729/512   3/2   128/81   27/16   16/9   243/128   2

(Numbers in red denote notes not belonging to the Pythagorean scale (C-scale)). Including the tonic, they consist of seven *pairs* that, as intervals, span an octave. In order of increasing magnitude, they are:

1.	1	2	1×2 = 2
2.	256/243	243/128	256/243×243/128 = 2
3.	9/8	16/9	9/8×16/9 = 2
4.	32/27	27/16	32/27×27/16 = 2
5.	81/64	128/81	81/64×128/81 = 2
6.	4/3	3/2	4/3×3/2 = 2
7.	1024/729	729/512	1024/729×729/512 = 2

The 14 notes consist of two pairs of Pythagorean notes, four pairs of a Pythagorean and a non-Pythagorean note and one pair of non-Pythagorean notes. The significance of this structure will emerge later.

As demonstrated in Article 16[5], every one of the 28 intervals between two notes belonging to each of the seven different musical scales is equal to the tone ratio of one of the 13 different notes above the tonic that can be found in these scales. In other words, all their intervals are just repetitions of this basic set.

Table 3 lists the tone ratios of the eight D–D' scales. They comprise 30 Pythagorean

notes and 18 non-Pythagorean notes. Table 4 compares the number of notes in the

Table 3. The tone ratios of the notes of the eight D–D' musical scales.

Musical scale	Tone ratio							
D' scale	1	9/8	32/27	4/3	3/2	27/16	16/9	2
C scale	1	9/8	81/64	4/3	3/2	27/16	243/128	2
B scale	1	256/243	32/27	4/3	1024/729	128/81	16/9	2
A scale	1	9/8	32/27	4/3	3/2	128/81	16/9	2
G scale	1	9/8	81/64	4/3	3/2	27/16	16/9	2
F scale	1	9/8	81/64	729/512	3/2	27/16	243/128	2
E scale	1	256/243	32/27	4/3	3/2	128/81	16/9	2
D scale	1	9/8	32/27	4/3	3/2	27/16	16/9	2

(White cells denote Pythagorean notes; red cells denote non-Pythagorean notes).

C–C' scales and the D–D' scales. Both have the same numbers of notes (24) and their octave complements (24). It is straightforward to verify that all of the other scales except the F scale (Lydian mode) exhibit this symmetry possessed by the C scale (Pythagorean) and D scale (Dorian musical mode). The latter is unique among the scales in that its pattern of intervals — TLTTTLT — is its own mirror reflection.

Table 4. Numbers of different notes in the eight C–C' and the eight D–D' scales

C scale–C' scale		D scale–D' scale	
6(9/8)	5(16/9)	6(9/8)	6(16/9)
4(81/64)	3(128/81)	3(81/64)	3(128/81)
7(4/3)	7(3/2)	7(4/3)	7(3/2)
4(32/27)	5(27/16)	5(32/27)	5(27/16)
2(256/243)	3(243/128)	2(256/243)	2(243/128)
1024/729	729/512	1024/729	729/512

Total = 24   Total = 24   Total = 24   Total = 24

The tonic, octave and the 24 pairs of notes and their complements can be assigned to the **50** corners of the first (6+6) regular polygons of the inner Tree of Life (Fig. 7). Article 4 showed that this set of polygons constitute a Tree of Life pattern because the number values of the ten Godnames define its geometrical properties.[6] The two endpoints of the shared root edge denote the tonic and the octave, and a corner and its mirror image in

50 =

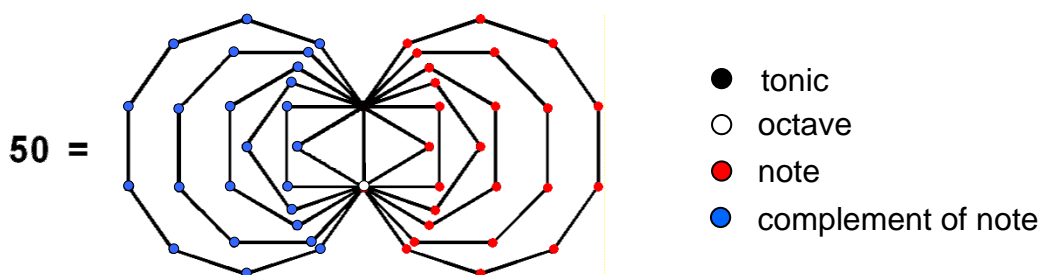● tonic
○ octave
● note
● complement of note

Figure 7. The first (6+6) polygons constitute a Tree of Life pattern. The endpoints of the root edge denote the tonic and octave, the 24 external corners on one side denote 24 notes of the eight D–D' scales between the tonic and octave and their mirror images denote their complementary notes.

the other similar set of polygons denote a note and its complement.

Suppose that the eight notes of a scale are denoted by points on a great circle of a sphere passing through the South Pole (tonic) and North Pole (octave) (Fig. 8). As the intervals of the E, F & G scales are the mirror images of, respectively, the C, B & A scales, a note is diametrically opposite every other note on the circle. As the pattern of
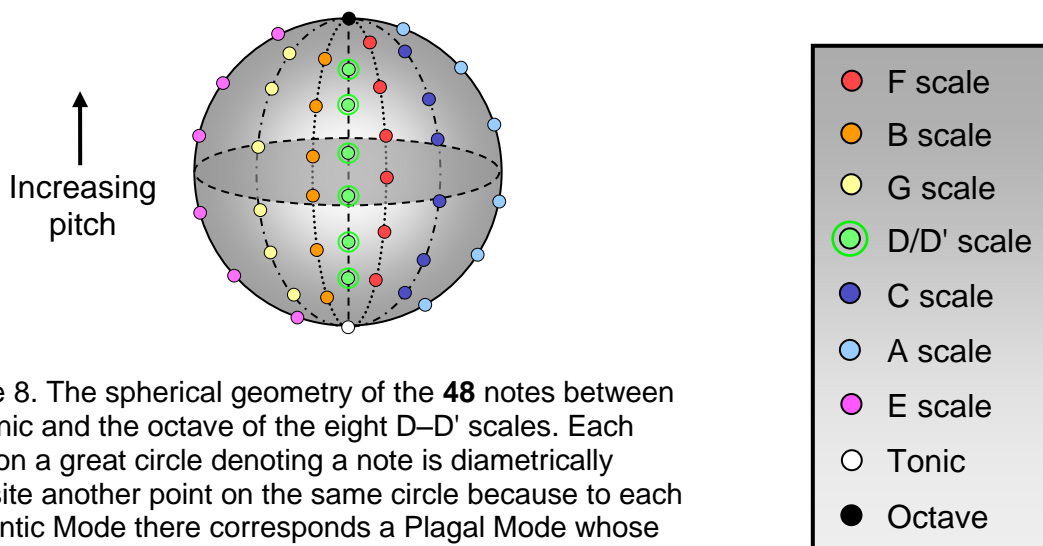

Figure 8. The spherical geometry of the **48** notes between the tonic and the octave of the eight D–D' scales. Each point on a great circle denoting a note is diametrically opposite another point on the same circle because to each Authentic Mode there corresponds a Plagal Mode whose pattern of intervals is its mirror image. The notes of the D and D' scale lie along the vertical axis of the sphere.

intervals of the D scale is its own mirror image, its notes can be represented by points on the central vertical axis of the sphere, each point being the same distance from the centre as its mirror image. This means that the eight scales D–D' can be represented by (2+8×6 = **50**) notes on the sphere and its axis, 12 of them lying on the latter as six different pairs of identical notes assigned to the six points between the North and South Poles. It demonstrates how the Godname ELOHIM with number value **50** prescribes this sequence of scales. It also demonstrates the power of the Tetrad (4) to express properties of holistic systems such as the seven musical scales, for the number of points symbolising their notes is 44. The Divine Name ELOHA assigned to Geburah with number value **36** prescribes the (6×6=**36**) notes between the tonic and the octave of the E–C scales that lie on the surface of the sphere and surround the 14 notes on its axis.

The 42 notes between the tonics and octaves of the seven different musical scales consist of the following pairs:

$$
\begin{array}{llll}
A = 5 & \times 5 \longrightarrow 5(9/8) & 5({\color{red}16/9}) & 9/8 \times {\color{red}16/9} = 2 \\
I = 10 & \times 10 \left\{ \begin{array}{l} 6(4/3) \\ 4({\color{red}32/27}) \end{array} \right. & \begin{array}{l} 6(3/2) \\ 4(27/16) \end{array} & \begin{array}{l} 4/3 \times 3/2 = 2 \\ {\color{red}32/27} \times 27/16 = 2 \end{array} \\
H = 5 & \times 5 \left\{ \begin{array}{l} 3(81/64) \\ 2({\color{red}256/243}) \end{array} \right. & \begin{array}{l} 3({\color{red}128/81}) \\ 2(243/128) \end{array} & \begin{array}{l} 81/64 \times {\color{red}128/81} = 2 \\ {\color{red}256/243} \times 243/128 = 2 \end{array} \\
A = 1 & \times 1 \longrightarrow {\color{red}1024/729} & {\color{red}729/512} & {\color{red}1024/729 \times 729/512} = 2
\end{array}
$$

The Godname EHYEH (AHIH) with number value **21** prescribes the **21** notes and their **21** complements, its letter values specifying different combinations of notes. As pointed out earlier, the Godname YAHWEH prescribes the **26** Pythagorean notes in this set of 42 notes.

The **50** notes of the eight scales D–D' represented in Fig. 8 consist of seven sets of

notes and their seven sets of complementary notes:

$$25 \begin{cases} 1(1) & 1(2) \\ 6(9/8) & 6(\textcolor{red}{16/9}) \\ 3(81/64) & 3(\textcolor{red}{128/81}) \\ 7(4/3) & 7(3/2) \\ 5(\textcolor{red}{32/27}) & 5(27/16) \\ 2(\textcolor{red}{256/243}) & 2(243/128) \\ \textcolor{red}{1024/729} & \textcolor{red}{729/512} \end{cases} 25$$

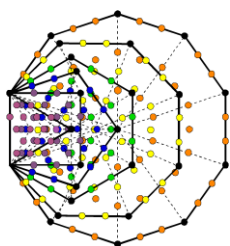There have to be at least seven pairs of complementary notes because there are seven notes in a scale below the octave. The fact, however, that the eight scales contain *only*

Figure 9. Associated with each set of six enfolded polygons are **168** yods other than their **26** corners. They symbolise the **168** intervals between the notes of the eight D–D' scales above the tonic.

seven pairs is non-trivial, because, although they comprise 14 different notes when the tonic is included, there is no obvious reason why they should group into seven pairs of complementary notes. It becomes obvious only when all possible intervals between the notes in each scale are calculated and found to consist of the *same* set of tone ratios of their notes.[7] This is because every interval has its complementary interval and therefore 14 different intervals must be grouped into seven pairs. As there is a note in the seven scales whose tone ratio equals one of these intervals, all their types of notes can be grouped into seven pairs.

As pointed out earlier, the D scale is uniquely symmetric in its sequence of intervals. It is also the only scale whose **36** intervals have the values of **26** Pythagorean tone ratios.[8] This prescription by the Godnames ELOHA with number value **36** assigned to Geburah and YAHWEH with number value **26** assigned to Chokmah reflects its status in Church music as the Dorian Mode, labelled Mode 1 to indicate that it is the scale from which all the other scales (ending with Mode 7, the mathematically most harmonious Pythagorean scale) can be generated by shifting one interval at a time. Furthermore, the D scale is the only one with 10 intervals with non-Pythagorean values.[9] As the seven octave species have 66 non-Pythagorean intervals,[10] the eight D-D' scales have (66+10=**76**) non-Pythagorean intervals. This is how the Godname YAHWEH ELOHIM with number value **76** prescribes the set of eight Church musical modes: it measures all its non-Pythagorean intervals.

Excluding the interval 1 between a note and itself, the seven different scales D–C have 130 Pythagorean intervals.[11] Excluding their seven octaves, they have 123 Pythagorean intervals. The eight notes of the D scale have **26** Pythagorean intervals. They include eight '1's and one '2,' that is, (**26**–8–1=17) such intervals between 1 and 2. The number of intervals between 1 and 2 in the eight scales D–D' = 123 +17 + **76** = **216**, which is the number value of Geburah. **48** of these are their notes between their tonic and octave, *leaving 168 intervals between the notes of the eight D–D' scales above the tonic*. This is the number of *Cholem Yesodoth*, Mundane Chakra of Malkuth. It is the superstring structural parameter discussed in many previous articles. They are symbolised in the first six enfolded, regular polygons as the **168** yods within them other than their **26** corners (Fig. 9), which, as we saw in Fig. 7, symbolise notes of the eight

scales. They are symbolised in the spherical representation of the eight D–D' scales as the **168** intervals between the seven notes above the tonic in each scale.

We found in Article 16[12] that the seven scales D–C have 84 rising, Pythagorean intervals that are repetitions of the six Pythagorean notes between the tonic and the octave. They have 84 falling intervals as their counterparts, making **168** in total. They are symbolised by the 84 yods along the edges of the first six enfolded polygons and

84 (●) 84 (●)
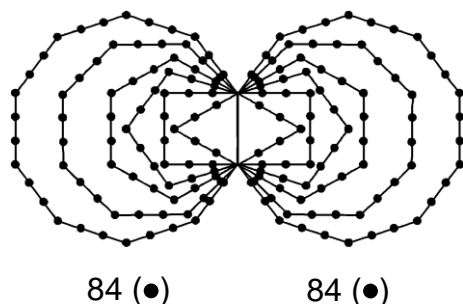
Figure 10. **168** yods lie along the edges of the first (6+6) polygons outside their shared edge. They symbolise the 84 Pythagorean rising intervals and their 84 falling counterparts (reciprocals) that are repetitions of the Pythagorean intervals in the seven musical scales.

their 84 mirror images in the other set of polygons (Fig. 10). Mirror symmetry of corresponding corners denotes the difference between a rising and a falling interval.

In terms of notes of the C scale, the composition of the 84 rising, Pythagorean intervals is

D	E	F	G	A
28	11	23	17	5

Therefore, there are (2×(28+11) = 78) rising and falling seconds and thirds and 2×(23+17+5) = 90 rising and falling fourths, fifths and sixths (the seventh never appears more than once in a scale). 78 is the number value of the Kabbalistic word 'Cholem' and 90 is the number value of the word 'Yesodoth.'

The seven different scales have 12 basic notes with different tone ratios between 1 and 2. All the intervals between notes in these scales have values found in this set. Hence, there are in the eight scales (**216**–12=204) repetitions of these notes and intervals. Including the tonic and the octave, the **50** notes of these scales arranged as in Fig. 8 and starting with the same note have 206 notes/intervals other than the basic set. Of these, 136 are Pythagorean notes/intervals and 70 are non-Pythagorean notes/intervals. It was shown in Article 29[13] that the triakis tetrahedron, the simplest Catalan solid, has 136 vertices, edges and triangles surrounding its centre when its internal triangles are regarded as single tetractyses and its faces are divided into three tetractyses. There are as many geometrical elements surrounding this polyhedron as there are Pythagorean notes and intervals in the eight scales in addition to its basic set. The reason for this correspondence is that both the eight D–D' musical scales and the triakis tetrahedron conform to the archetypal pattern of the Tree of Life, as proved for the latter in Article 29, and therefore must display properties that numerically correspond to each other.

Ten Pythagorean notes (tonic, octave & two identical sets of the four notes E, G, A & B of the D scale) lie on the central axis of the spherical representation of the eight D–D' scales. Hence, the 206 intervals consist of (136–10=126) Pythagorean intervals of the six scales other than the two D scales and **80** intervals made up of 70 non-Pythagorean intervals and 10 Pythagorean notes arranged along the axis. As, according to Table 2 of Article 29, the triakis tetrahedron has 132 geometrical elements surrounding its axis, six of which are its vertices, 126 other vertices, edges and triangles surround its axis.

Once again, correspondence appears between different features of each system.

The counterpart in the triakis tetrahedron of the fact discussed above that **168** intervals exist between the notes above the tonic of the eight D–D' scales is that, when its internal triangles are divided into three tetractyses, it has **168** geometrical elements surrounding its axis. As the Pythagoreans taught, "music is geometry."

As commented upon earlier, the 14 different notes of the seven musical scales consist of seven pairs of notes and their complements:

1.	1	2	1×2 = 2
2.	256/243	243/128	256/243×243/128 = 2
3.	9/8	16/9	9/8×16/9 = 2
4.	32/27	27/16	32/27×27/16 = 2
5.	81/64	128/81	81/64×128/81 = 2
6.	4/3	3/2	4/3×3/2 = 2
7.	1024/729	729/512	1024/729×729/512 = 2

(The ordering is in terms of increasing size of the first member of each pair). Let us write, firstly, the octave 2 as '**1**' (using boldface to remind ourselves that **1** is not the number 1), secondly, each note of one set as a_i (i = 1–7) and, thirdly, its complement as
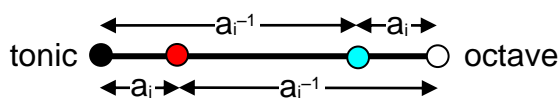
Figure 11. A note (●) and its complement (●) are equidistant from the ends of a musical scale.

a_i^{-1}. The product of an interval and its complement may then be expressed as

$$a_i a_i^{-1} = 1.$$

The complement of a given note is as far below the octave as that note is above the tonic — they are equally spaced from each end of the scale (Fig. 11). The complement can be regarded as the inverse of a note in the sense that their intervals span the octave. A rising musical interval a = n (n>1) has a complement with interval 2/n, *not* 1/n, which is the value of the falling interval. Every member of the set of 14 notes has a complement, or inverse, which is also a member of that set. As the D scale is unique in having a pattern of intervals that is identical to its mirror image, this means that it alone amongst the scales is made up of four pairs of notes and their complements:
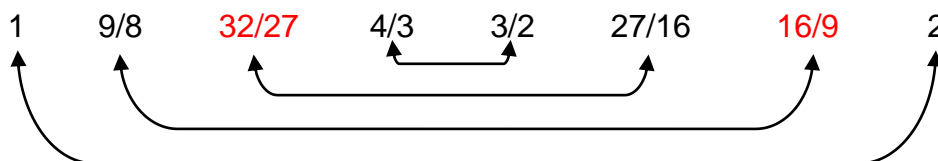
In terms of the Pythagorean tone T and leimma L, the seven pairs of notes making up the musical scales have the form:

1.	1	T^5L^2 (= 2)
2.	L	T^5L
3.	T	T^4L^2
4.	TL	T^4L
5.	T^2	T^3L^2
6.	T^2L	T^3L
7.	T^2L^2	T^3

Eight notes are Pythagorean and six notes (written above in red) are non-Pythagorean. By arranging the intervals between successive notes of the Pythagorean scale on a circle (Fig. 12), all possible scales are the sequences of seven intervals generated by
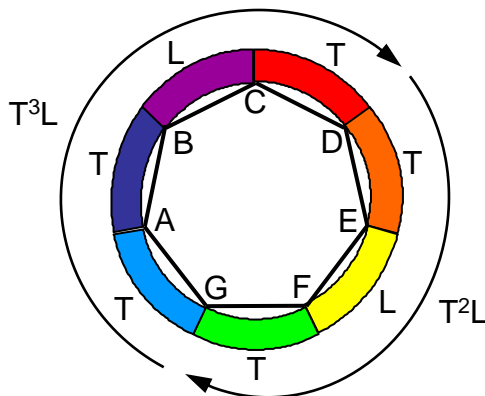

Figure 12. The seven musical scales are the seven possible consecutive sequences of intervals arranged on a circle. The 13 different intervals between their notes are all possible, different combinations of *consecutive* intervals.

starting at successive intervals until all of them have been selected. Their notes and intervals are all possible different combinations of *consecutive* intervals. They have the forms listed above. The complement of any note represented by an arc of the circle in Fig. 12 is that created by the opposing arc that completes the circular pattern of intervals. Just as every colour has its complementary colour, so, too, do the notes.

The eight notes of a musical scale have $\binom{8}{2}$ = 28 intervals, of which seven are those of the seven notes above the tonic. There are **21** intervals between these notes (Fig. 13). The eight D–D' scales have (**21×8=168**) such rising intervals between their (8×7=56)
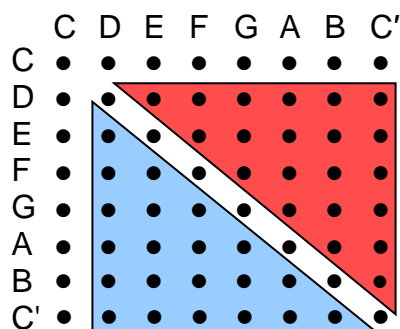
Each of the notes of the eight D–D' musical scales (the C scale is shown here) has **21** rising intervals (denoted by dots in the red triangle) between notes other than their tonics and **21** falling intervals (denoted by dots in the blue triangle). The eight scales have (8×**21**=**168**) rising intervals and **168** falling intervals.

Figure 13. The 56 notes of the eight D–D' musical scales above their tonics have **168** rising intervals and **168** falling intervals.

notes. Similarly, there are **168** falling intervals. They are symbolised in the inner Tree of Life by the **168** yods in each set of the first six enfolded polygons (Fig. 14), whose corners denote the **50** notes of the eight D–D' scales arranged on the surface of the sphere and along its axis, as discussed earlier and shown in Fig. 8.

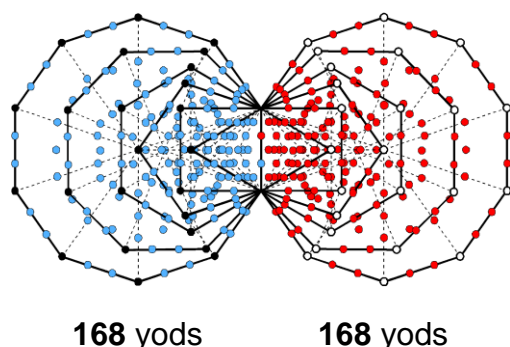**168** yods   **168** yods

Figure 14. The (6+6) enfolded polygons have **50** corners symbolising the **50** notes of the eight musical scales arranged on a sphere. Associated with each set of six polygons are **168** yods other than their corners. One set of yods symbolises the **168** rising intervals between their notes. The other set symbolises their **168** falling intervals.

This remarkable analogy between the intervallic composition of the eight church musical modes and the inner Tree of Life would remain valid whatever set of eight scales was considered. The eight scales starting with the D scale have been chosen because: 1. it was the Dorian mode that was, historically speaking, made the first of the Authentic Modes, and 2. starting with the D scale means that the Pythagorean scale is the last different scale that can be generated from it — a sequence that is natural in view of its special significance as the most mathematically harmonious scale and its historical importance as the basis of Western musical theory. The interval composition of the scales does, of course, depend upon which set of eight scales is considered.

The seven pairs of notes and their complements consist of two pairs of Pythagorean notes (tonic/octave & perfect fourth/fifth) and five pairs of notes that each includes at least one non-Pythagorean note. The 2:5 division is another example of the basic
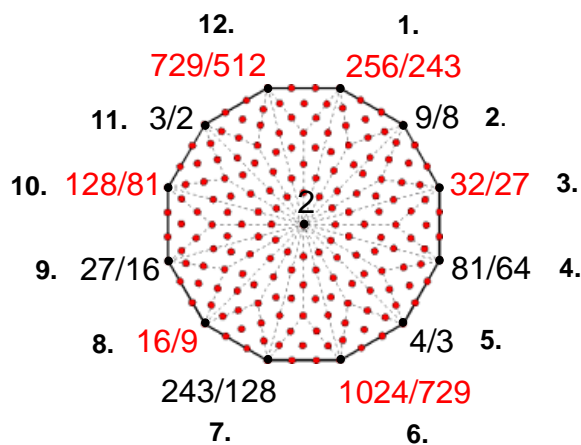
Figure 15. The 12 corners of the dodecagon symbolise the 12 different notes between the tonic and the octave of the eight D–D' scales. A note and its complementary note are diametrically opposite each other. The central yod denotes the octave and the **168** (●) yods surrounding it denote the **168** intervals between the notes above the tonic.

pattern that starts in the seven scales, each with five tone intervals T with magnitude 9/8 and two leimmas L with magnitude 256/243.

The last of the regular polygons in the inner Tree of Life is the dodecagon. As the tenth regular polygon, it represents the completion of the Pythagorean paradigm of wholeness symbolised by the tetractys. When its 12 sectors are each divided into three tetractyses, it contains 181 yods (Fig. 15). 13 yods are at the corners and centre of the dodecagon and **168** yods are generated by this transformation. The 12 types of musical
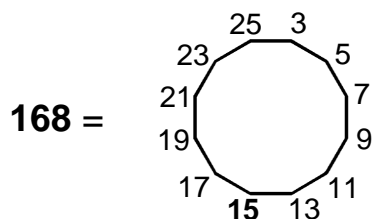

Figure 16. The sum of the first 12 odd integers after 1 is the number of intervals between the notes above the tonic of the eight D–D' musical scales.

notes between the tonic and the octave in the seven types of scales can be assigned to its corners, with the octave assigned to its centre. The **168** rising intervals between them can be assigned to the **168** yods. The **168** yods in the other dodecagon symbolise the **168** falling intervals. This demonstrates the remarkable connection between the sacred geometry of the inner Tree of Life and the musical potential of the eight scales.

As

$$13^2 - 1 = \mathbf{168} = 3 + 5 + 7 + \ldots + 25,$$

168 is not only the number of yods in a dodecagon other than its centre and corners but also the sum of the first 12 odd integers that can be assigned to its corners (Fig. 16).

This shows the arithmetic, geometric and musical connection between the numbers 12 and **168**. It appears in the first four Platonic solids ending with the icosahedron with 12 vertices as the average number of yods needed to build them out of tetractyses.[14]

Inspection of Tables 3, 4, 5, 7, 8 & 9 in Article 16[15] shows that the C, E, F, G, A & B scales have (including octaves) **112** Pythagorean intervals and 56 non-Pythagorean intervals, that is, **168** intervals. Once again, the number value of *Cholem Yesodoth* measures the number of intervals — this time for the six scales other than the D scale and including octaves.

2. Connection between the 144 Polyhedron and the eight scales
We found earlier that the eight D–D' scales have **216** intervals other than octaves between their **64** notes. We also found that they have 12 types of notes other than the octave and **168** intervals between notes above the tonic, totalling 180 intervals other

144 Polyhedron

216 edges → **216** intervals other than octaves between
64 notes of eight D–D' scales

Disdyakis triacontahedron

180 edges → 12 basic notes in eight D–D' scales
between tonic and octave + **168**
intervals between notes above the tonic
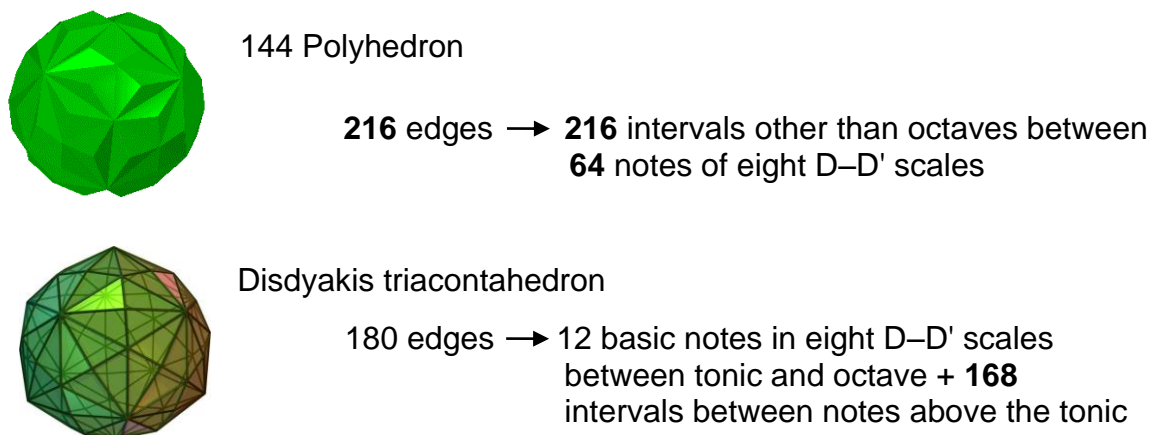
Figure 17. The edges of the 144 Polyhedron & disdyakis triacontahedron represent all the possible intervals below the octave between the notes of the eight Church musical modes.

than the octave. Compare this with the facts that the 144 Polyhedron has **216** edges and that the disdyakis triacontahedron has 180 edges (Fig. 17). In the first case, there are (8×6 + **168** = **216**) intervals. The (8×6=**48**) notes between the tonic and octave of the eight scales contain the bas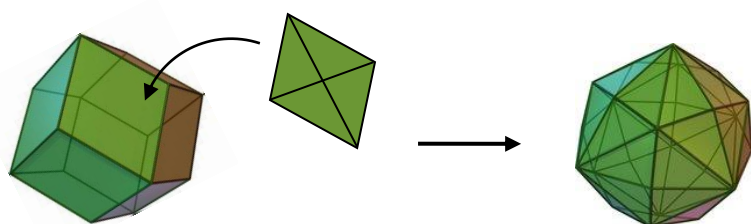ic set of 12 notes, that is, there are **36** repetitions of some of these notes, where **36** is the number value of ELOHA, Godname of Geburah. These repetitions are represented by the **36** extra edges of the 144 Polyhedron. We see that the edges of the 144 Polyhedron represent all possible intervals other than octaves between the notes of the

Figure 18. Attaching pyramids to the 12 faces of a rhombic dodecahedron generates the rhombic triacontahedron.

eight Church musical modes and that the edges of the disdyakis triacontahedron represent the basic set of 12 intervals and their **168** repetitions:

144 Polyhedron: **216 = 48 + 168**.

Disdyakis triacontahedron: 180 = 12 + **168**.

The disdyakis dodecahedron that generates the 144 Polyhedron is itself generated from the rhombic dodecahedron by attaching pyramids to its 12 rhombic faces (Fig. 18). This increases the number of vertices from 14 to (14+12=**26**), the number of edges from 24 to (24+12×4=**72**) and the number of faces from 12 to (12×4=**48**). The change in the number of edges: 24→(24+48=72)→(**72**+144=**216**), has its counterpart in the eight Church modes, whose **216** intervals below the octave comprise the **48** notes between the tonic and octave, (8×3=24) intervals between the octave and the second, third and

72 edges
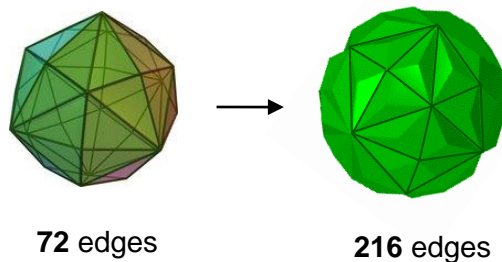
216 edges

The **216** intervals other than octaves between the notes in the 8 D-D' scales comprise **48** note intervals, (8×3=24) intervals between the octave and the 2nd, 3rd & 4th notes and 144 other intervals.

Figure 19. The **216** edges of the 144 Polyhedron include the **72** edges of the underlying disdyakis dodecahedron. Similarly, the **216** intervals in the 8 D-D' scales include the **72** intervals that are either notes or intervals between the octave and the 2nd, 3rd & 4th notes.

fourth notes in each scale, and 144 other notes (Fig. 19). Notice that the **216** intervals split also into the **48** notes and **168** other intervals, where **168** is the number value of *Cholem Yesodoth*, the Mundane Chakra of Malkuth and a basic structural parameter of the E₈×E₈ heterotic superstring. The number **48** is represented in the 144 Polyhedron as the **48** peaks of the tetrahedra that have to be added to the faces of the disdyakis dodecahedron in order to create it. The 24 notes and their 24 complements correspond to the 24 peaks and their 24 mirror images. The latter originate in the four faces of a pyramid that is attached to each of the (6+6) faces of the rhombic dodecahedron. This means that each set of 24 peaks is six sets of four. Similarly, the 24 notes and their 24 complements can be regarded as six sets of eight notes, one note from each of the four Authentic Modes and one note from each of the four Plagal Modes.

The **48** notes are copies of the basic set of 12 notes between the tonic and octave of the seven types of musical scales:

256/243   9/8   32/27   81/64   4/3   1024/729   729/512   3/2   128/81   27/16   16/9   243/128

Eight notes comprise four pairs of Pythagorean notes and their non-Pythagorean complements. These four pairs can be assigned to eight of the 12 vertices of an icosahedron denoting these notes (Fig. 20). These are the 12 B vertices of the
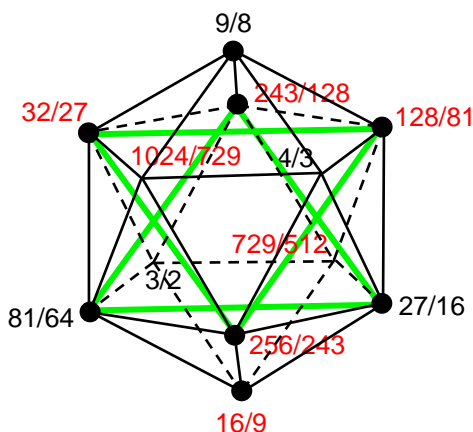

Figure 20. The 12 notes between the tonic and octave of all the musical scales can be assigned to the vertices of an icosahedron. Diametrically opposite vertices denote a note n and its complement m, where nm = 2. Green lines connect eight vertices (●) of an icosahedron representing the four pairs of Pythagorean notes and their non-Pythagorean complements.

disdyakis triacontahedron. The eight vertices are those of an octahedron.[1]

Including the tonic and the octave, the seven scales contain eight different Pythagorean notes and six different non-Pythagorean notes. The former cannot be assigned to an octahedron because they do not consist of four pairs of notes and their complements. This means that its vertices *must* denote four pairs of notes and their complements, where each pair is a Pythagorean note and a non-Pythagorean note. The four other notes comprise a pair of Pythagorean notes (perfect fourths & fifths) and a pair of non-Pythagorean notes:

$$
8 \begin{cases} 256/243 & 243/128 & 256/243 \times 243/128 = 2 \\ 9/8 & 16/9 & 9/8 \times 16/9 = 2 \\ 32/27 & 27/16 & 32/27 \times 27/16 = 2 \\ 81/64 & 128/81 & 81/64 \times 128/81 = 2 \end{cases}
$$
$$
4 \begin{cases} 4/3 & 3/2 & 4/3 \times 3/2 = 2 \\ 1024/729 & 729/512 & 1024/729 \times 729/512 = 2 \end{cases}
$$

Each diametrically opposite pair of B vertices denotes a note and its complement. The inversion symmetry of the icosahedron has its musical counterpart in the fact that the complement of any note in the seven different musical scales with tone ratio n is one with tone ratio 2/n that is also one of these notes. Notice that the four Pythagorean notes that can be assigned to four vertices of an octahedron: 9/8, 27/16, 81/64 & 243/128, are each separated by a perfect fifth. Similarly for their respective complements 16/9, 32/27, 128/81 & 256/243. The four remaining B vertices of the icosahedron signify the pair of Pythagorean notes (4/3, 3/2) & the pair of non-Pythagorean notes (1024/729, 729/512) at the *centre* of the basic set of 12 different notes making up the seven different musical scales:

256/243 9/8 32/27 81/64 | 4/3 1024/729 729/512 3/2 | 128/81 27/16 16/9 243/128

The 30 Pythagorean notes of the eight D–D' scales between the tonic and the octave are assigned to the 30 A vertices of the disdyakis triacontahedron and its 18 non-

16 △

48 vertices
48 sides
16 triangles

10 vertices $(10=1+2+3+4)$
22 sides $(22=1^4+2^3+3^2+4^1)$
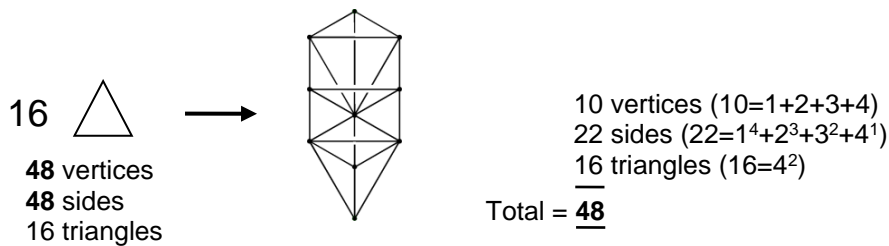16 triangles $(16=4^2)$

Total = **48**

Figure 21. 16 triangles come together and join at 22 sides to create the Tree of Life with **48** vertices, edges & triangles.

Pythagorean notes, the tonic and the octave are assigned to the 20 C vertices of the dodecahedron. The musical correspondence is summarised below:

Eight D–D' musical scales		Disdyakis triacontahedron
30 Pythagorean notes	→	30 A vertices
12 different notes of 7 types of musical scales	→	12 B vertices
Tonic, octave and 18 non-Pythagorean notes	→	20 C vertices
180 intervals between notes of eight scales	→	180 edges
120 intervals between **48** notes between	→	120 edges above faces of

---

[1] It may be relevant that an octahedron with eight faces has **48** rotational and mirror symmetries.

tonic and octave of eight scales	rhombic triacontahedron
12 basic intervals + **48** intervals between octave and six notes between tonic and octave	$\longrightarrow$ 60 edges of rhombic triacontahedron

3. Comparison with the Tree of Life

Given that previous articles have demonstrated that the disdyakis triacontahedron is the polyhedral, outer form of the inner Tree of Life, it should come as no surprise that other aspects of this form should appear in the musical context, as now explained. 16

Tree of Life

10 vertices
22 edges
16 triangles

Total = **48**

= **48** =

trunk

10 vertices
10 edges
5 triangles
1 tetrahedron

Total = **26**

= **26** =

26 =

Y	H	V	H
10	5	6	5

$10(\bullet)$ $[4(-) + \triangledown] + 6(-) + [4(\triangledown) + \triangledown]$
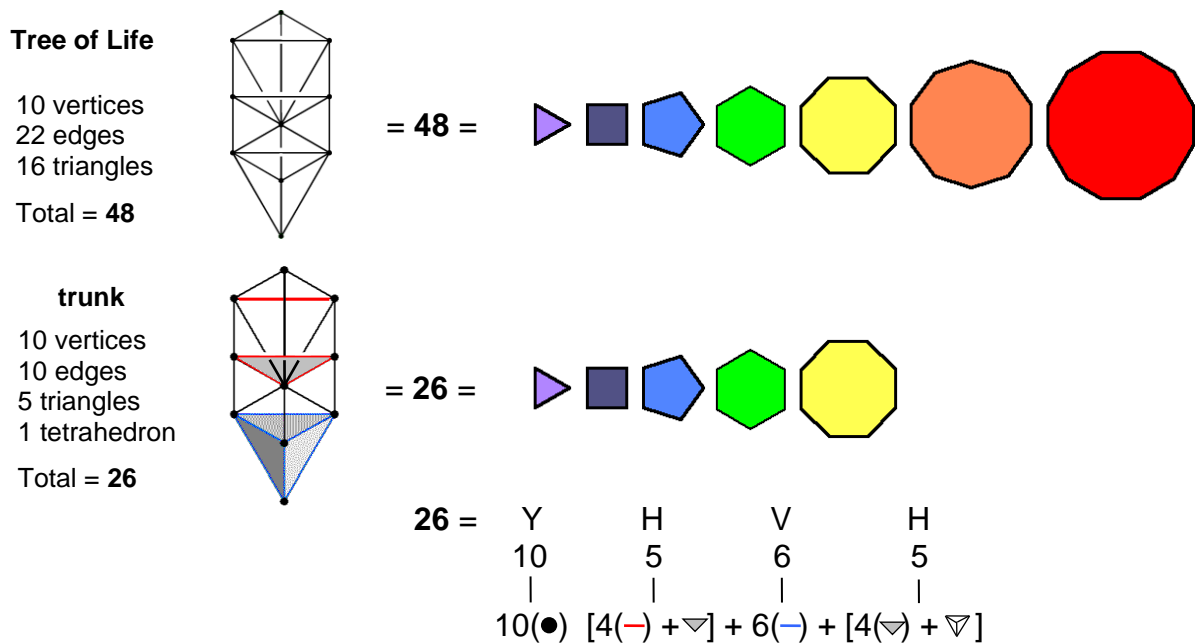
Figure 22. The outer Tree of Life is composed of **48** vertices, edges & triangles that correspond to the **48** corners of the seven regular polygons constituting its inner form. YAHWEH with number value **26** prescribes the **26** geometrical elements forming the trunk of the outer tree. Its letter values denote different groups of geometrical elements. YAHWEH also prescribes the first five polygons of the inner tree with **26** corners.

separate triangles combine by joining 41 of their edges to create the outer form of the Tree of Life with 10 vertices of 16 triangles with 22 edges (Fig. 22). **64** geometrical elements (38 vertices, **26** edges) disappear in this union, where **64** = 4^3. It comprises **48** geometrical elements made up of 10 vertices, 22 edges and 16 triangles. As the geometrical representation of the integers 1, 2, 3 & 4, the 'trunk' of the Tree of Life consists of the point (Kether), line (Chokmah-Binah), triangle (Chesed-Geburah-Tiphareth) and tetrahedron (Netzach-Hod-Yesod-Malkuth). Fig. 22 shows that it is
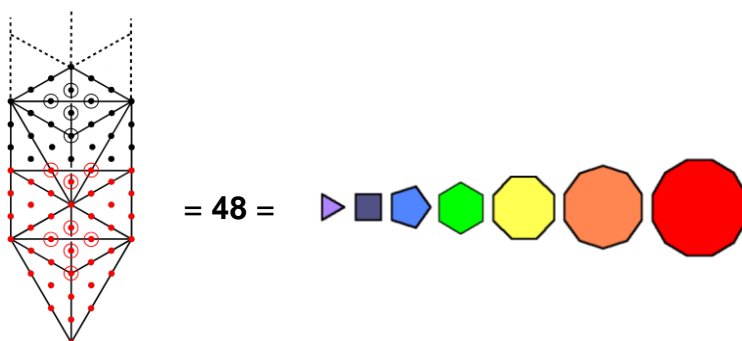

= **48** =

Figure 23. The seven separate, regular polygons comprising the inner Tree of Life have as many corners (**48**) as there are (●) yods in the 1-tree up to the level of Chesed, the first Sephirah of Construction.

Tone ratios of the 14 different notes in the seven musical scales

1	L	T	TL	T^2	T^2L	T^2L^2	T^3	T^3L	T^3L^2	T^4L	T^4L^2	T^5L	T^5L^2
1	256/243	9/8	32/27	81/64	4/3	1024/729	729/512	3/2	128/81	27/16	16/9	243/128	2

Archimedean solid	falling interval	rising interval	Catalan solid
truncated tetrahedron	243/256	256/243	triakis tetrahedron
cuboctahedron	8/9	9/8	rhombic dodecahedron
truncated cube	27/32	32/27	triakis octahedron
truncated octahedron	64/81	81/64	tetrakis hexahedron
rhombicuboctahedron	3/4	4/3	deltoidal icositetrahedron
snub cube	729/1024	1024/729	pentagonal icositetrahedron
icosidodecahedron	512/729	729/512	rhombic triacontahedron
truncated cuboctahedron	2/3	3/2	disdyakis dodecahedron
truncated icosahedron	81/128	128/81	triakis icosahedron
truncated dodecahedron	16/27	27/16	pentakis dodecahedron
rhombicosidodecahedron	9/16	16/9	deltoidal hexacontahedron
snub dodecahedron	128/243	243/128	pentagonal hexacontahedron
truncated icosidodecahedron	1/2	2	disdyakis triacontahedron

Figure 24. The 13 Catalan solids and their 13 duals (excluding their chiral versions) are the polyhedral counterpart of the 13 different rising and 13 different falling intervals between the notes of the seven types of musical scales. The counterpart of the perfect fifth with tone ratio 3/2 (the midpoint of the Pythagorean scale) is the disdyakis dodecahedron with **48** faces, which generates the 144 Polyhedron. The central member of the set of 13 notes above the tonic has tone ratio 729/512. Its counterpart is the rhombic triacontahedron with 30 faces, which generates the disdyakis triacontahedron.

made up of **26** geometrical elements. This trunk is prescribed by the Godname YAHWEH with number value **26**, its letter values denoting different groups of elements. The counterpart of the **48** elements in the inner form of the Tree of Life is the **48** corners of the seven regular polygons. The counterpart of the trunk in the inner Tree of Life is the set of the first five polygons, which have **26** corners.

Transformed into tetractyses, the 19 triangles of the lowest Tree of Life of any set of overlapping Trees of Life contain **80** yods. There are **48** yods up to the level of the path joining Geburah to Chesed — the first Sephirah of Construction (Fig. 19). This further demonstrates that **48** fundamental degrees of freedom are needed to express just that part of the Tree of Life that contains the seven Sephiroth of Construction. This number is the basic structural parameter of the Tree of Life — whether geometrical elements in the case of its outer form or corners of polygons in the case of its inner form. It
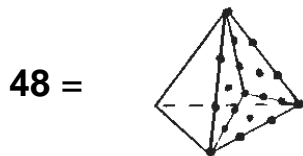
48 =

Figure 25. Division of each face of a tetrahedron into three tetractyses generates **48** hexagonal yods (only one face so constructed is shown).

quantifies whatever holistic system is designed according to this blueprint. In the case of the eight Church musical modes, the number **48** is the number of notes between the tonic and octave. It should therefore come as no surprise that it figures so prominently in the construction of the geometry of the disdyakis triacontahedron. We saw earlier that, if the 12 notes of the eight musical scales are assigned to the corners of the dodecagon, the extra **168** yods needed to construct it from 12 tetractyses symbolise the **168** intervals between their notes. Thus formed, its centre is surrounded by 12 vertices, 24 edges and 12 triangles, that is, **48** geometrical elements are needed to construct the dodecagon, starting from a point. Once again, the number **48** defines the *minimal* number of bits of information required to create a complete Tree of Life pattern.

When its faces are divided into three tetractyses, the simplest Platonic solid — the tetrahedron — has **48** hexagonal yods in its 12 tetractyses (Fig. 25). The basic building block of polyhedra is composed of **48** yods that symbolise the seven Sephiroth of Construction. Any system whose structure conforms to the universal blueprint of the Tree of Life must necessarily possess this characteristic number of degrees of freedom.

We found in Article 27 that, when its interior triangles and its faces are each divided into three triangles, the disdyakis triacontahedron comprises 2400 vertices, edges & triangles.[16] As

$$49^2 - 1 = 3 + 5 + 7 + \ldots + 97,$$

this number is the sum of the first **48** odd integers after 1. Yet again, the number **48** defines arithmetically the very geometrical composition of the disdyakis triacontahedron — the polyhedral manifestation of the inner Tree of Life.

Just as the rhombic triacontahedron underlies the disdyakis triacontahedron, so the Catalan solid with the next larger number of faces — the disdyakis dodecahedron — is the foundation of the 144 Polyhedron, which is the result of adding tetrahedra to its **48** faces with **26** vertices. In fact, it may be seen as the 'trunk' of this polyhedron. Just as YAHWEH (YHVH) with number value **26** prescribes the trunk of the Tree of Life because the latter comprises **26** geometrical elements (Fig. 22) and the disdyakis dodecahedron because it has **26** vertices, so YAH (YH) with number value **15** marks out this member of the family of **26** Archimedean and Catalan solids. This is because,

starting from the simplest Catalan solid — the triakis tetrahedron — and counting back and forth between pairs of dual polyhedra, the disdyakis dodecahedron is the **15**th polyhedron (see Fig. 24), whilst, starting from the simplest Archimedean solid, the **26**th
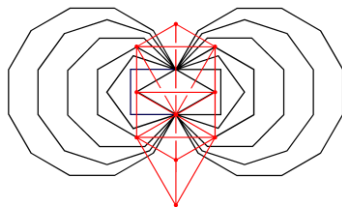

Figure 26. Of the 70 corners of the two sets of 7 regular polygons, eight are shared with the outer Tree of Life. The **62** unshared corners symbolise new degrees of freedom manifesting in 3-dimensional space as the **62** vertices of the disdyakis triacontahedron.

solid is the disdyakis triacontahedron, excluding chiral partners. The Godname YAHWEH therefore determines *both* polyhedra in the polyhedral Tree of Life whilst YAH prescribes the family of **15** Archimedean solids, the family of **15** Catalan solids and the disdyakis dodecahedron as the **15**th in the two families, including their chiral versions.

When the outer Tree of Life is projected onto the plane containing its inner form (Fig. 26), eight of the 70 corners of the two sets of enfolded polygons coincide with the
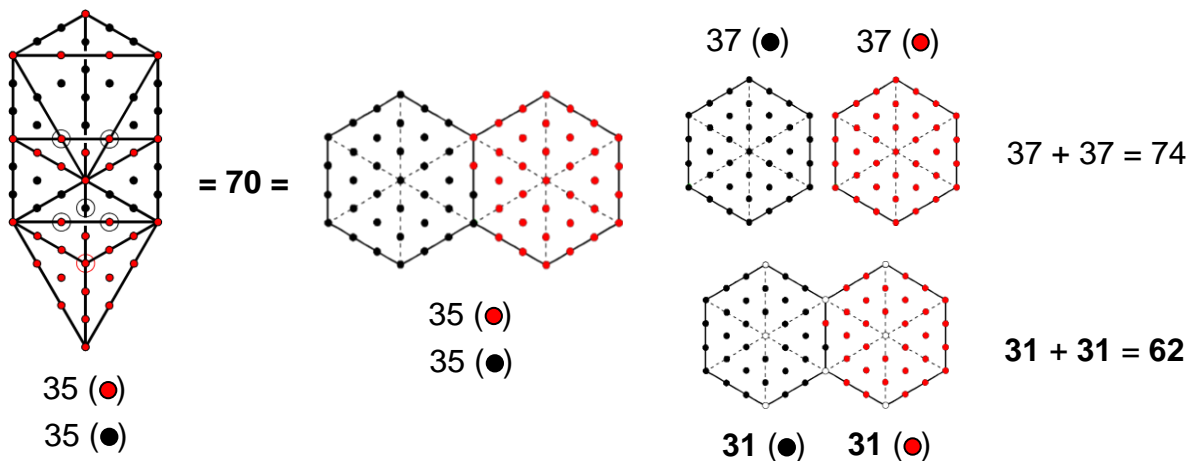
Figure 27. The pair of hexagons in the inner Tree of Life has as many yods as the Tree of Life, the 35 yods in its trunk being the number of yods associated with each polygon. The two separate hexagons contain 74 yods symbolising the 74 vertices of the 144 Polyhedron. The two joined hexagons have **62** yods other than the eight (O) yods shared with the Sephiroth of the Tree of Life. These unshared yods symbolize the **62** vertices of the disdyakis triacontahedron.

projections of Sephiroth and Daath. The **62** unshared corners represent new degrees of freedom. They are the **62** vertices of the disdyakis triacontahedron.

The hexagon is the fourth regular polygon. Constructed from tetractyses, a pair of separate hexagons is made up of 74 yods (Fig. 27). They symbolise the 74 vertices of the 144 Polyhedron. Its **48** 'new' vertices correspond to the **48** yods on the boundaries of the pair of hexagons and at the centres of their sectors, whilst its **26** other vertices (vertices of the underlying disdyakis dodecahedron) correspond to the **26** radial yods. Joined together in the inner Tree of Life, the pair of hexagons has **62** yods other than those shared with the outer Tree of Life. They symbolise the **62** vertices of the disdyakis triacontahedron. The 12 yods at the centres of each sector of the hexagons denote its 12 B vertices, the 20 yods lying on their boundaries symbolise the 20 C vertices and the 30 radial yods symbolise its 30 A vertices.

The isomorphism between the pair of hexagons and the disdyakis triacontahedron

provides a natural representation of the 12 different notes in the seven musical scales because we found earlier that they correspond to the 12 B vertices of the icosahedron. The six yods at centres of sectors in one hexagon (Fig. 28) denote (in order of increasing tone ratios) the six notes 256/243, 9/8, 32/27, 81/64, 4/3 & 1024/729 and

30 (○) → 30 A vertices

12 (○) → 12 B vertices

20 (●) → 20 C vertices

○ yod coincides with a Sephirah of the outer Tree of Life.

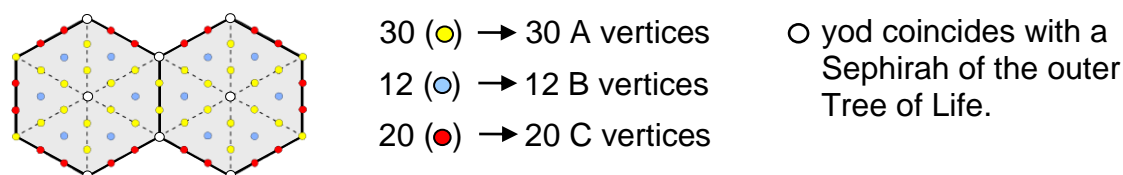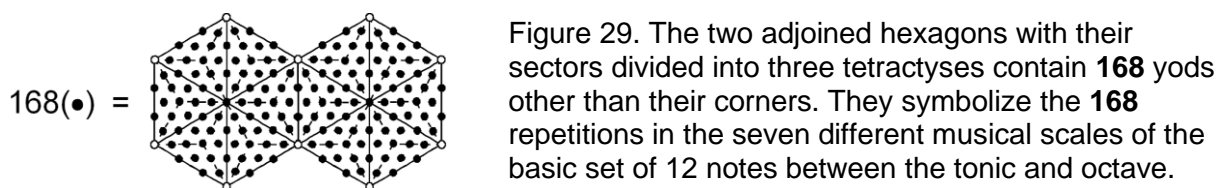Figure 28. The 30 radial (○) yods symbolize the 30 A vertices of the disdyakis triacontahedron, the 12 (○) yods at centres of sectors symbolize its 12 B vertices and the 20 (●) yods on the external boundaries denote its 20 C vertices.

their mirror images in the other hexagon denote the complements of these notes. The 30 Pythagorean notes in the eight D–D' scales are symbolised by radial yods and the 20 notes consisting of the tonic, octave and 18 non-Pythagorean notes are represented by the 20 boundary yods.

That this parallelism between the pair of hexagons and the disdyakis triacontahedron is not merely due to coincidence is indicated by the fact that they require **168** more yods to construct their sectors from three tetractyses (Fig. 29). As discussed earlier (see Fig.

$168(\bullet) =$

Figure 29. The two adjoined hexagons with their sectors divided into three tetractyses contain **168** yods other than their corners. They symbolize the **168** repetitions in the seven different musical scales of the basic set of 12 notes between the tonic and octave.

10), this is the number of rising and falling, *repeated* intervals between the notes between the tonic and the octave of the seven different types of musical scales.[17] The 84 yods associated with one hexagon denote their 84 rising intervals and the 84 yods in the other hexagon denote their 84 falling intervals.

4. The holistic character of the number 33

The **62** vertices of the disdyakis triacontahedron are arranged in either seven sheets perpendicular to an A-A axis, 11 sheets perpendicular to a B-B axis or **15** sheets perpendicular to a C-C axis. The correspondence between this pattern and the note composition of the seven different musical scales was mapped out in Article 26.[18] It was pointed out in Section 1 that they comprise **26** Pythagorean notes between the tonic and the octave and seven octaves, that is, 33 Pythagorean notes. The tonal composition of the former is

$$\mathbf{26} = 5(9/8) + 6(3/2) + 3(81/64) + 2(243/128) + 6(4/3) + 4(27/16)$$

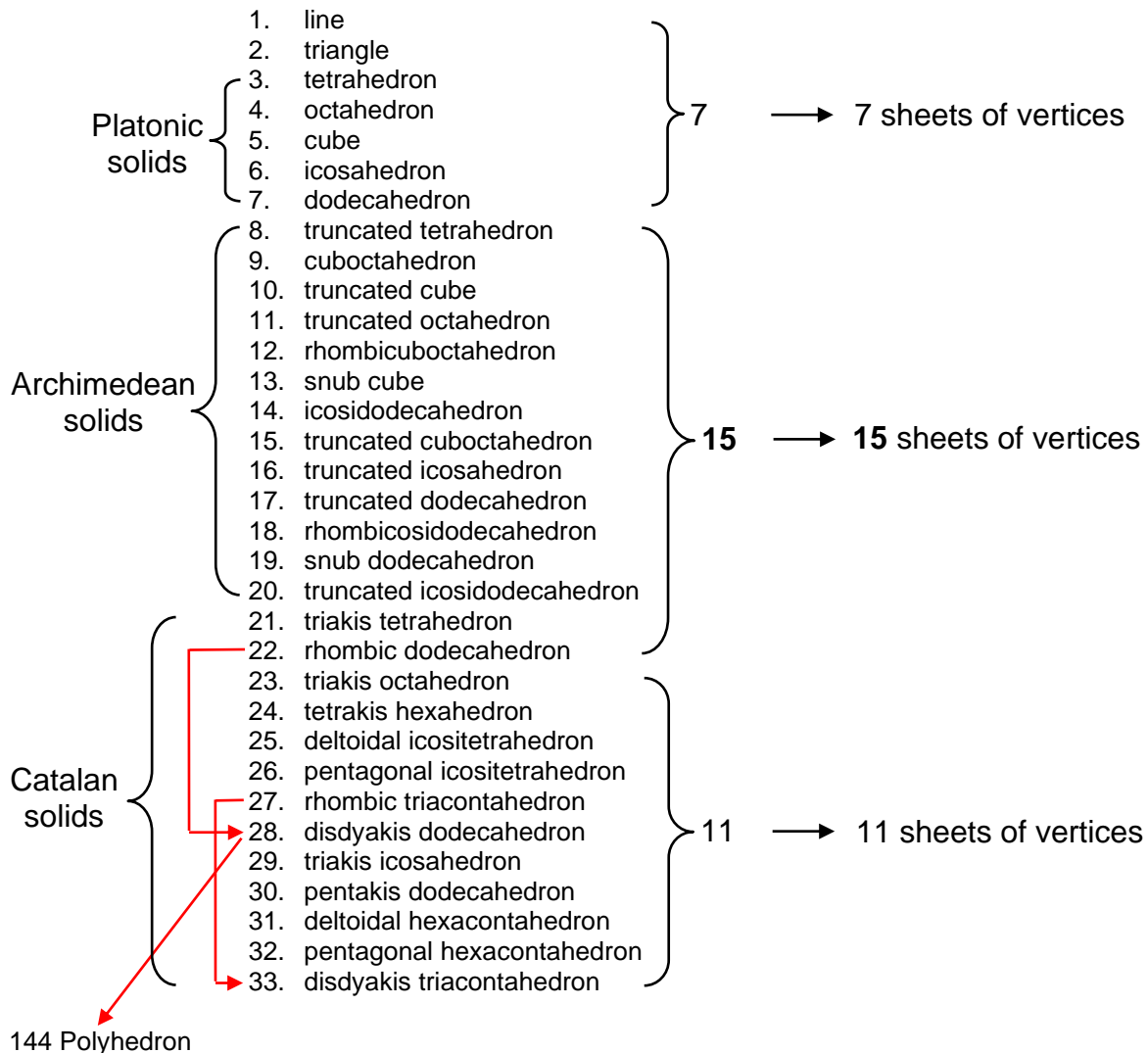| | 5 | 6 | | 5 | | 10 |
| | H | V | | H | | Y |

| 11 | | 15 |

The letter values of YAHWEH denote the numbers of notes of one or two types.

Alternatively, the **26** Pythagorean notes between the tonic and the octave may be divided up in terms of the seven musical scales/church modes listed in Table 1:

Mode 1	D scale	4		
Mode 2	A scale	3		
Mode 3	E scale	2	10 = Y	
Mode 4	B scale	1		**15** = YH
Mode 5	F scale	5	H	
Mode 6	C scale	6	V	11
Mode 7	G scale	5	H	

However the composition of the **26** Pythagorean notes is decided, the crucial point is that the 7:**15**:11 pattern of sheets of vertices is *identical* to the pattern of different notes in the seven different scales, the seven sheets of vertices corresponding to their seven octaves. The parallelism exists because both systems — one geometrical, one musical — are Tree of Life patterns that are prescribed by Divine Names such as YAH and YAHWEH. The number 33 is always a measure of a *whole*.

The same pattern can also be seen in the building up of the regular and semi-regular polyhedra. The first stage of creation of form is the straight line, the second stage is the triangle and the third stage is the tetrahedron, the simplest of the five regular polyhedra, or Platonic solids. The dodecahedron, the last of the Platonic solids, is formed at the seventh stage and the disdyakis triacontahedron, the last of the **26** semi-regular polyhedra, represents the 33rd stage:

1. line
2. triangle
3. tetrahedron
4. octahedron
5. cube
6. icosahedron
7. dodecahedron

Platonic solids — 7 ⟶ 7 sheets of vertices

8. truncated tetrahedron
9. cuboctahedron
10. truncated cube
11. truncated octahedron
12. rhombicuboctahedron
13. snub cube
14. icosidodecahedron
15. truncated cuboctahedron
16. truncated icosahedron
17. truncated dodecahedron
18. rhombicosidodecahedron
19. snub dodecahedron
20. truncated icosidodecahedron

Archimedean solids — **15** ⟶ **15** sheets of vertices

21. triakis tetrahedron
22. rhombic dodecahedron
23. triakis octahedron
24. tetrakis hexahedron
25. deltoidal icositetrahedron
26. pentagonal icositetrahedron
27. rhombic triacontahedron
28. disdyakis dodecahedron
29. triakis icosahedron
30. pentakis dodecahedron
31. deltoidal hexacontahedron
32. pentagonal hexacontahedron
33. disdyakis triacontahedron

Catalan solids — 11 ⟶ 11 sheets of vertices

144 Polyhedron

Listed above are, firstly, the Archimedean solids and then the Catalan solids. Just as it exhibits 33 sheets of vertices, so the disdyakis triacontahedron is the 33rd stage in the generation of the semi-regular polyhedra. The **15**th stage after the seventh generates the rhombic dodecahedron with **26** vertices & faces, **36** edges & faces and **50** vertices, edges & faces. With its geometry prescribed by four Godnames (YAH = **15**, YAHWEH = **26**, ELOHIM = **50** and ELOHA = **36**), it should not be surprising that this polyhedron plays a fundamental role in the generation of the 144 Polyhedron. Sticking rhombic pyramids to its 12 faces creates the disdyakis dodecahedron with **48** faces and **26** vertices (shown above linked by a red arrow to the rhombic dodecahedron). It is *this* polyhedron that is fundamental to the generation of the 144 Polyhedron because it generates the latter in a way analogous to how the rhombic triacontahedron creates the disdyakis triacontahedron. Sticking tetrahedra to its **48** faces creates the 144 Polyhedron with (**48+26**=74) vertices and (3×**48**=144) faces, whilst adding pyramids to the faces of the rhombic triacontahedron generates the disdyakis triacontahedron.

It was shown in Article 24[19] that the geometry of the Tree of Life generates the rhombic faces of the rhombic dodecahedron and the golden rhombic faces of the rhombic triacontahedron.

Confirmation that the rhombic dodecahedron is, as the **15**th stage in the generation of the disdyakis triacontahedron, the polyhedral root of the polyhedral Tree of Life is provided by the letter values of YAHWEH. In the last 11 stages, the rhombic triacontahedron is the fifth stage and the disdyakis dodecahedron is the sixth stage and the first of the last five stages. The rhombic dodecahedron is picked out by the first two letters of YHVH and the two polyhedra that determine the two polyhedra in the polyhedral Tree of Life are selected by the number values of the last two letters H and V of YHVH:

1. triakis octahedron
2. tetrakis hexahedron
3. deltoidal icositetrahedron
4. pentagonal icositetrahedron
5. rhombic triacontahedron

$\left.\right\}$ 5 = H

6. disdyakis dodecahedron
7. triakis icosahedron
8. pentakis dodecahedron
9. deltoidal hexacontahedron
10. pentagonal hexacontahedron
11. disdyakis triacontahedron

$\left.\right\}$ 6 = V

Notice also that the disdyakis dodecahedron is **21**st in the list of semi-regular polyhedra and so is prescribed by the EHYEH, Godname of Kether with number value **21**. The **21**st stage marks the triakis tetrahedron, the simplest of the Catalan solids. It was shown in Article 29[20] that, if the holistic disdyakis triacontahedron is likened to the tetractys — the Pythagorean symbol of holistic systems — the triakis tetrahedron is its 'yod.' This is because the geometrical properties of the former are exactly *ten* times the corresponding one for the latter. The triakis tetrahedron is built from 137 vertices, edges & triangles, where 137 is the 33rd prime number and the number known to physicists to define the fine structure constant $e^2/\hbar c \approx 1/137$. The number 33 not only determines the stage of generation of the disdyakis triacontahedron — the last Catalan solid — but also the geometrical composition of the first Catalan solid, whose properties are multiplied tenfold in the latter!

The reason why 33 appears in the Pythagorean note composition of the seven musical scales and in the mathematical description of the disdyakis triacontahedron is that both

are holistic systems whose components are 33 in number. Or, better, that they are manifestations of a universal cycle that requires 33 stages to complete itself. This is demonstrated par excellence in the tree level structure of ten overlapping Trees of Life,

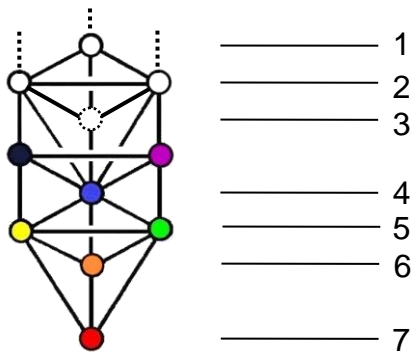Figure 30. The seven tree levels.

each representing one of the ten Sephirah of the Tree of Life. Fig. 30 indicates that the latter has seven horizontal divisions marking the stages of emanation of Sephiroth of Construction. Although Daath is not a Sephirah, this point in the Tree of Life is Yesod of the next higher, overlapping tree and therefore counts as a division unless there is no higher tree. These divisions are called 'tree levels.' Fig. 31 shows that ten overlapping Trees of Life have 33 tree levels (in general, n trees have (3n+3) tree levels). They measure the complete cycle of emanation of 10 Sephiroth. In an abstract sense, the number 33 denotes the number of stages in the completion of a cycle of development of a system that culminates in an object that fully embodies the universal blueprint of the Tree of Life. The relationship between the 33 vertex sheets of the disdyakis triacontahedron and the 33 tree levels of 10 overlapping Trees of Life was analysed in Article 25.[21] Also discussed there is what
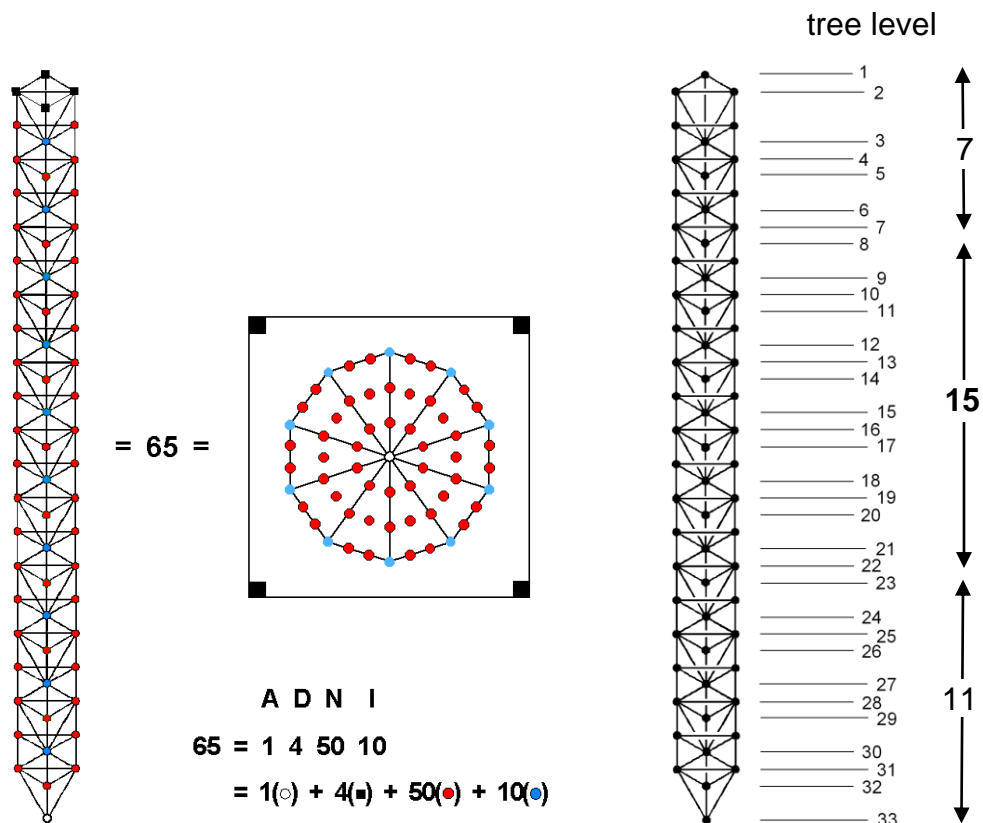
Figure 31. Equivalence of the 10-tree prescribed by the Godname ADONAI and its tetractys representation. Ten overlapping Trees of Life have 33 tree levels.

the 7:**15**:11 vertex sheet composition of the disdyakis triacontahedron means for superstring physics. The 10-dimensions of space-time that are predicted by superstring theory are mapped by 10 overlapping Trees of Life. The Godname ADONAI (ADNI)

10
tetrahedra

5 cubes

5 octahedra

5 rhombic
dodecahedra

1 dodecahedron

1 icosahedron

1 rhombic
triacontahedron

Figure 32. The 28 polyhedra making up the disdyakis triacontahedron.
(Images from: http://www.rwgrayprojects.com/Lynn/P120BUP/buildup.html)

assigned to Malkuth prescribes the lowest 10 trees because its number value **65** is the number of their Sephirothic emanations. Fig. 31 shows how the letter values of ADNI specify different types of emanations. The number 33 appears here because **65** is t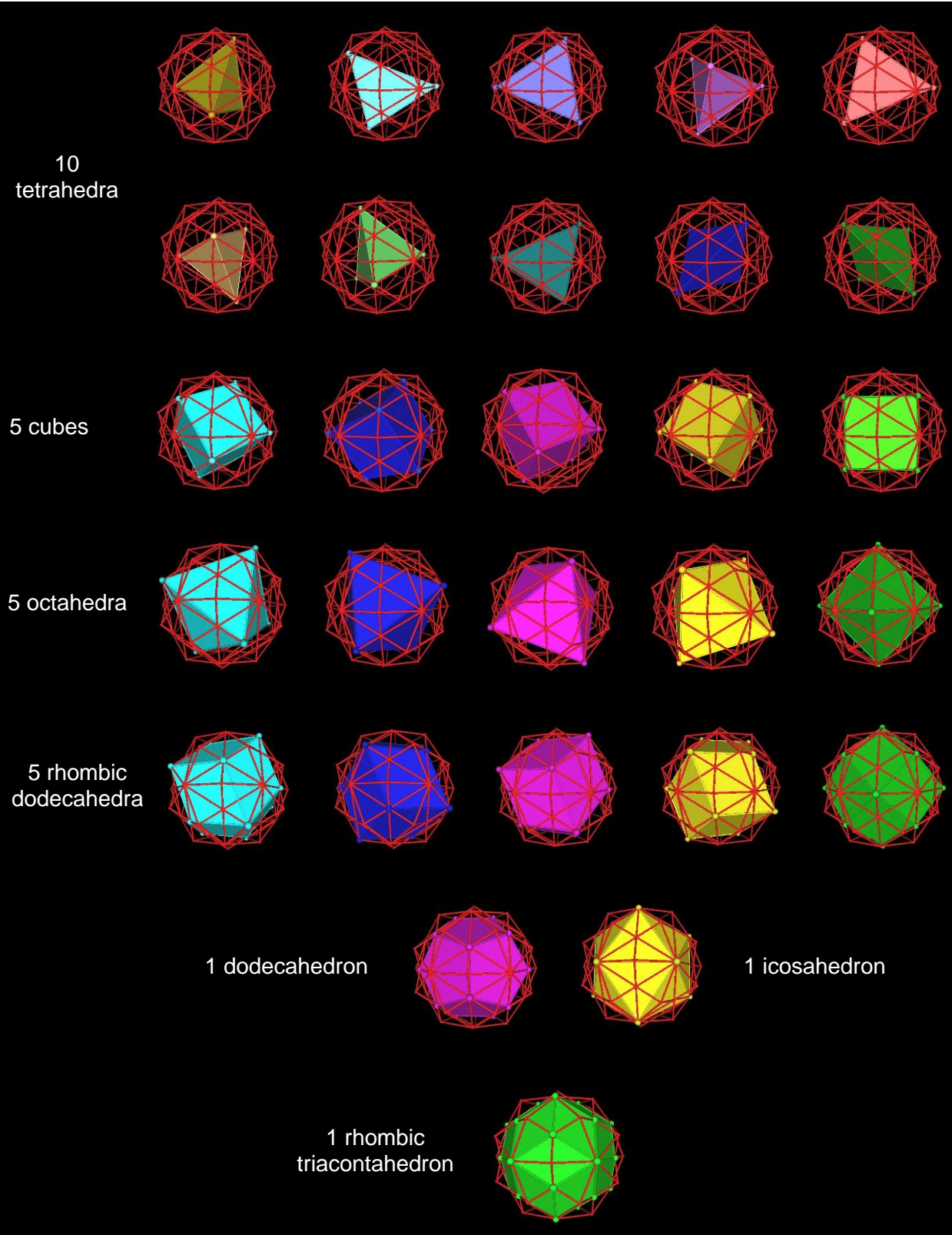he 33rd odd integer after 1. Once again, it quantifies a holistic system, because each tree represents one of the Sephiroth.

Counting from the tonic of the first octave of the Pythagorean scale, the tone ratio 24 is the 33rd note and the perfect fifth of the fifth octave (Table 5). Counting from the latter, the 33rd note is $576 = 24^2$ and still the perfect 5th of the new fifth octave. This is the **65**th note from the tonic of the first octave. The Godname Adonai with number value **65**

Table 5. Every 33rd note in the Pythagorean scale increases in pitch by a factor of 24.

	C	**D**	**E**	**F**	**G**	**A**	**B**	**Number of overtones**
1	1	9/8	81/64	4/3	3/2	27/16	243/128	0
2	2	9/4	81/32	8/3	3	27/8	243/64	2
3	4	9/2	81/16	16/3	6	27/4	243/32	4
4	8	9	81/8	32/3	12	27/2	243/16	7
5	16	18	81/4	64/3	24	27	243/8	11
6	32	**36**	81/2	128/3	**48**	54	243/4	**15**
7	64	**72**	81	256/3	96	108	243/2	20
8	128	144	162	512/3	192	**216**	243	**26**
9	256	288	324	1024/3	384	432	486	32
10	512	576	648	2048/3	768	864	972	38
11	1024	1152	1396	4096/3	1536	1728	1944	39

prescribes sequences of 33 notes in the Pythagorean scale whose last note has a tone ratio always 24 times that of the first note. This does not depend on which note is the starting point. Any sequence of 33 notes ends in one whose frequency is 24 higher than that of the first, thus demonstrating the cyclic character of this number.

The significance of the perfect 5th of the fifth octave is that it is the last of *eight* successive fifths (not all perfect fifths):

G_1	D_2	A_2	E_3	B_3	F_4	C_5	G_5
3/2	9/4	27/8	81/16	243/32	32/3	16	24

(Subscripts denote the octave number)

In general, every 33rd note is every eight successive fifth. Table 5 indicates that the 33rd note is the *tenth* overtone. 22 of the notes up to the perfect 5th of the fifth octave are fractional. They comprise 16 notes in the first three octaves (separated from the rest of the table by a thick line) and six notes in the fourth and fifth octaves up to the last fifth. This 6:16 division corresponds in the Tree of Life to the six Paths that are edges of the tetrahedron whose vertices are the four lowest Sephiroth and to the 16 Paths outside it (Fig. 33). The 32 fractional notes and overtones up to G_5 conform to the geometrical pattern of the Tree of Life, the ten overtones corresponding to the ten

Sephiroth and the 22 fractional notes corresponding to the 22 Paths that connect them. The ordering of notes in Fig. 33 follows the traditional Kabbalistic numbering of Paths. As the tenth overtone, G_5 with tone ratio 24 corresponds to Malkuth, the lowest Sephirah, signifying the physical manifestation of any holistic system conforming to the
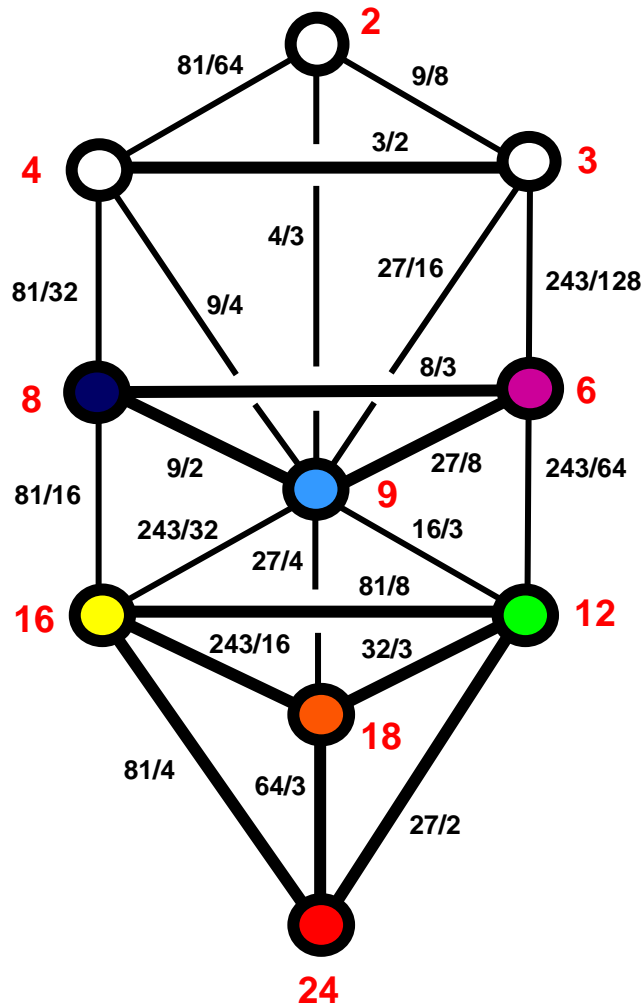
(Thick lines are Paths of the trunk of the Tree of Life)

Figure 33. The Tree of Life pattern of the first ten overtones.

Tree of Life blueprint. The significance of this number to superstring theory is discussed in Article 12.[22]

The number of corners of the 7n regular polygons enfolded in n overlapping Trees of Life is

$$C(n) = 35n + 1,$$

where "1" denotes the uppermost corner of the hexagon enfolded in the tenth tree (the hexagon is the only one of the seven regular polygons that shares a corner with polygons enfolded in adjacent trees. The shared corner is denoted by '1' in the above formula). Therefore, C(10) = **351** = 1 + 2 + 3 + …. + **26**. This shows how YAHWEH with number value **26** prescribes the inner form of 10 overlapping Trees of Life. **351** is the number value of *Ashim* ("Souls of Fire"), the Order of Angels assigned to Malkuth. The number of corners of the 7n polygons outside their n root edges = C'(n) – 2n = 33n + 1. C'(n+1) – C'(n) = 33. In other words, there are 33 corners per set of polygons outside

their root edge (Fig. 34). *The emanation of successive Trees of Life generates 33 new geometrical degrees of freedom associated with their inner form.* This again demonstrates the cyclic aspect of the number 33 in measuring the number of entities
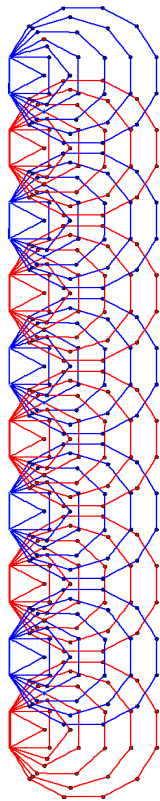

Figure 34. *Ashim*, the Order of Angels assigned to Malkuth, prescribes 10 overlapping Trees of Life because its number value is **351**, which is the number of corners of the 70 polygons enfolded in these trees. **351** = 1 + 2 + 3 +… + **26**, showing how the Divine Name YAHWEH with number value **26** prescribes the inner form of 10 Trees of Life. There are 34 corners in each set of seven polygons outside their shared edge. As the uppermost corner of the hexagon coincides with the lowest corner of the hexagon enfolded in the next higher tree, there are 33 corners *per set of polygons.* These independent, geometrical degrees of freedom associated with each tree correspond to the 33 tree levels of 10 overlapping Trees of Life.

33 (●) or (●) corners
per set of 7 polygons

(polyhedra, musical notes, etc) in a sequence that represents a holistic system conforming to the archetypal pattern of the Tree of Life.

When its triangles are turned into tetractyses, the number of yods in the lowest n overlapping Trees of Life is given by

$$Y(n) = \mathbf{50}n + 30.$$

Therefore, Y(33) = 1680. As discussed in many previous articles, this is the number of turns in each helical whorl of the UPA (Fig. 35), the indivisible unit of matter described by Annie Besant and C.W. Leadbeater 114 years ago with the aid of a yogic siddhi. In
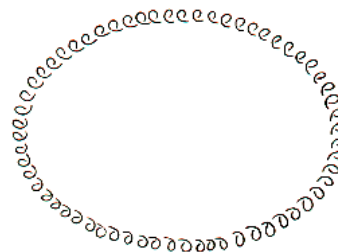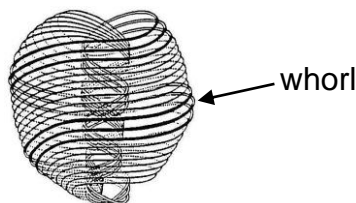
whorl

Each helical whorl
has 1680 coils.

UPA/heterotic superstring
(From *Occult Chemistry,* Annie
Besant & C.W. Leadbeater (1951)

Figure 35. Each of the 10 whorls of the UPA (identified by the author as the E$_8$×E$_8$ heterotic superstring constituent of a quark) is a helix with 1680 turns.

other words, the lowest 33 Trees of Life are made up of as many yods as there are circularly polarized waves in a whorl. This is how the number 33 determines the structural parameter of superstrings inside atomic nuclei. Malkuth of the 33rd Tree of Life is the **65**th emanation on the central Pillar of Equilibrium, showing how ADONAI, the Godname of Malkuth with number value **65**, prescribes this number.

As pointed out in Article 22,[23] there are seven steps leading to the generation of the disdyakis triacontahedron, starting with the tetrahedron. The first five are the Platonic solids and the sixth is the rhombic triacontahedron, the Catalan solid that is compounded from the dodecahedron and the icosahedron. When constructed from tetractyses, the numbers of hexagonal yods in their faces are listed below:

tetrahedron	**48**
octahedron	96
cube	96
icosahedron	240
dodecahedron	240
rhombic triacontahedron	240 + 240
disdyakis triacontahedron	240 + 240

$$\text{48, 96, 96} \} \text{240}$$

Total = 1680

As discussed earlier, the disdyakis triacontahedron is the 33rd stage in the cycle of development of regular and semi-regular solids. Once again, the number 33 is associated with the superstring structural parameter 1680. The five Platonic solids and the two semi-regular solids in this list are the polyhedral counterpart of the five whole intervals and two leimmas in the Pythagorean musical scale:

T   T   L   T   T   T   T   L

The **62** vertices of the disdyakis triacontahedron define the vertices of 28 regular and semi-regular solids: ten tetrahedra (two vertices at each C vertex), five cubes (two vertices at each C vertex), five octahedra (one vertex at each A vertex), one icosahedron (one vertex at each B vertex), one dodecahedron (one vertex at each C vertex), five rhombic dodecahedra (two vertices at each C vertex and one vertex at each A vertex) and one rhombic triacontahedron (vertices at B and C vertices). Constructed from tetractyses, the hexagonal yod populations of the faces of the 28 polyhedra definable within the disdyakis triacontahedron are:

tetrahedron:	10×**48** =	480
cube:	5×96 =	480
octahedron:	5×96 =	480
icosahedron:	1×240 =	240
dodecahedron:	1×240 =	240
rhombic dodecahedron:	5×192 =	960
rhombic triacontahedron:	1×480 =	480

1680 (first group), 1680 (second group)

Total = 3360

There are **21** Platonic solids of the four types that the ancient Greeks believed were the shapes of particles of the elements Earth, Water, Air and Fire. Their hexagonal yod population is 1680 — the same as the number of hexagonal yods in the seven polyhedra listed above. This is remarkable evidence of beautiful, mathematical design. EHYEH, the Divine Name associated with Kether at the top of the Tree of Life has the number value **21.** It prescribes this polyhedral embodiment of the superstring structural parameter 1680, its *letter values specifying the numbers of polyhedra of each type*:

A	H	I	H
1	5	10	5

icosahedron   octahedron   tetrahedron   cube
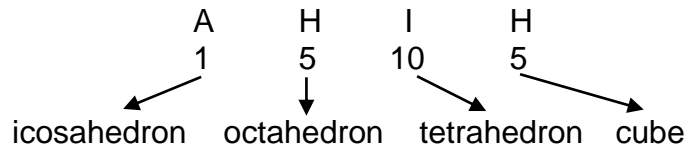
The remaining three types of polyhedra also have 1680 hexagonal yods in their faces, giving a total number of 3360. This is astounding for two reasons:

1. Divine Unity symbolised by the Pythagorean Monad, or mathematical point ("0th-order tetractys"), differentiates, firstly, into the familiar tetractys ("1st-order tetractys") with 10 yods (three corners, seven hexagonal yods), secondly, into the "2nd-order tetractys" with 85 yods (**15** corners, 70 hexagonal yods), and so on:
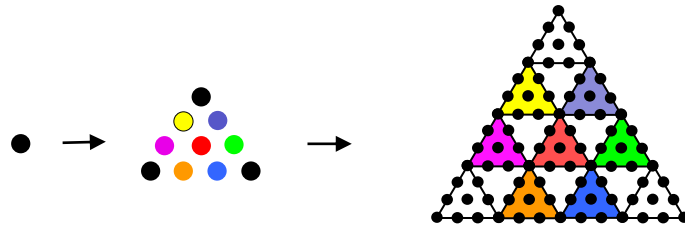
3360 is the number of yods in the seven enfolded, regular polygons constituting the inner form of the Tree of Life when their 47 sectors are each turned into 2nd-order tetractyses (Fig. 36).[24]

2. 3360 is the number of circularly polarised wave oscillations made during each of the five revolutions of the 10 whorls of the UPA (see Fig. 35). Each yod making up the seven enfolded polygons constructed from 2nd-order tetractyses symbolises one oscillation made in the 10 whorls during one complete revolution around the axis of the
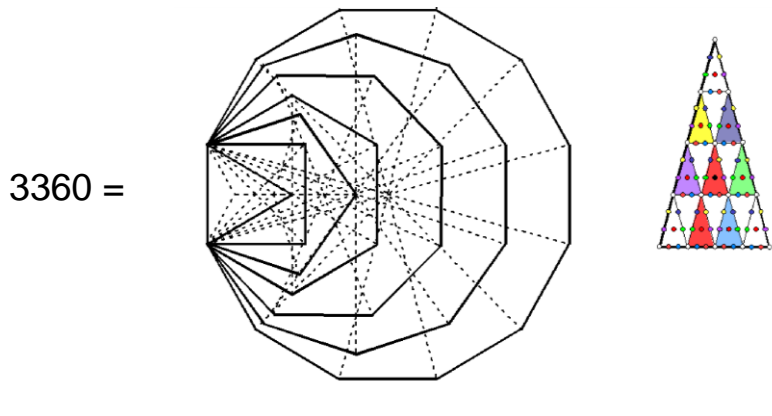

$$3360 =$$

Figure 36. Constructed from 2nd-order tetractyses, the seven enfolded regular polygons constituting the inner form of the Tree of Life contain 3360 yods. This is the number of hexagonal yods in the faces of the 28 regular and semi-regular solids generated by the **62** corners of the disdyakis triacontahedron.

UPA. This is not coincidence but a demonstration of the equivalence between the 2-dimensional form of the inner Tree of Life — the seven enfolded polygons — and its polyhedral form — the disdyakis triacontahedron.

Binah, the third Sephirah in the Tree of Life and the one embodying the cosmic feminine principle (Kabbalists call it Aima, the "Great Mother"), has a gematria number value of **67**. The number of yods in n overlapping Trees of Life is

$$Y'(n) = \mathbf{50}n + 20.$$

Therefore, $Y'($**67**$) = 3370$. There are 10 yods in the uppermost triangle whose base is the path joining Chokmah and Binah of the highest tree. Below Binah of the **67**th tree are 3360 yods. Amazingly, the hexagonal yod population of the 28 polyhedra within the disdyakis triacontahedron is the number of yods below Binah of the very number of trees that equals the number value of Binah!

5. The musical counterpart of G_2

The octonions form the fourth (and last) class of division algebras. An octonion has the form:

$$N = a_0e_0 + a_1e_1 + a_2e_2 + \ldots + a_7e_7,$$

where the a_i (i = 0–7) are real numbers, e_0 is the identity element and the seven unit octonions e_i (i = 1–7) are imaginary numbers: $e_i^2 = -1$. Their multiplication is anti-commutative:

$$e_ie_j = - e_je_i \hspace{4cm} (i{\neq}j)$$

non-associative:

$$e_i(e_je_k) \neq (e_ie_j)e_k$$

and follows the rule

$$e_ie_{i+1} = e_{i+3}.$$

Every unit imaginary octonion has an inverse, or conjugate, $e_i^* = -e_i$, where $e_ie_i^* = 1$. Similarly, the conjugate of an octonion N is N^*, where

$$N^* = a_0 - a_1e_1 - a_2e_2 - \ldots - a_7e_7.$$

The norm of an octonion is $||N|| = \sqrt{N^*N}$, where

$$||N||^2 = a_0^2 + a_1^2 + a_2^2 + \ldots + a_7^2.$$

The inverse of N is $N^{-1} \equiv N^*/||N||^2$, so that $N^{-1}N = NN^{-1} = 1$. It was shown in Article 28 that there are 13 classes of real, linear combinations of two or more unit octonions. It means that there are 13 classes of octonions that are inverses to the former, so that YAHWEH with number value **26** prescribes the **26** classes of octonions and their inverses.

This pattern is analogous to the 13 Archimedean solids and their duals — the 13 Catalan solids. The counterpart of polyhedral duality in the world of octonions is the
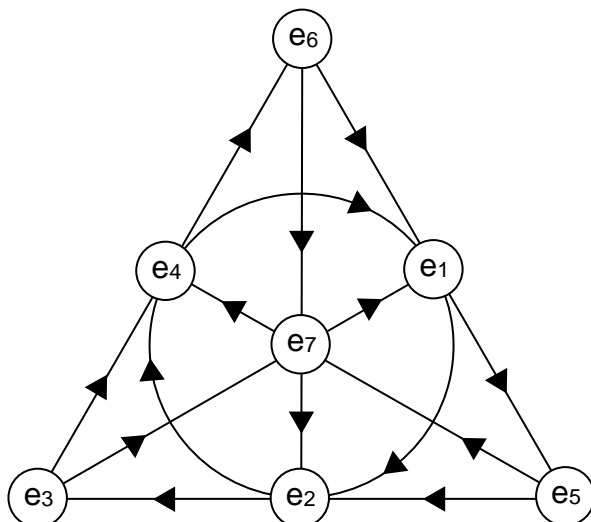

Figure 37. The Fano Plane represents multiplication of the unit imaginary octonions. Arrows connecting triplets of octonions indicate the order of multiplication that generates the third octonion on the same straight or curved line.

relationship of an octonion $N' = N^{-1}$ being the inverse of the octonion N, where N'N = 1. Its counterpart in music is the relationship between a note of tone ratio n and its complement of tone ratio m, where nm = 2. What is sufficiently remarkable in the analogy to take it seriously is that

1. just as there are 13 classes of octonions and 13 classes of their inverses, so the seven musical scales are made up of 13 different notes above the tonic, that is, they have 13 rising intervals and 13 falling intervals;

2. just as the seven imaginary octonions have seven inverses $e_i^* = -e_i$, so the 14 different notes of the seven scales are divided into seven notes and their complements:

imaginary octonion	inverse	musical notes	complement
e_1	e_1^{-1}	tonic (= 1)	octave T^5L^2 (= 2)
e_2	e_2^{-1}	L	T^5L
e_3	e_3^{-1}	T	T^4L^2
e_4	e_4^{-1}	TL	T^4L
e_5	e_5^{-1}	T^2	T^3L^2
e_6	e_6^{-1}	T^2L	T^3L
e_7	e_7^{-1}	T^2L^2	T^3

Moreover, just as the seven e_i form seven ordered sets of three, or 3-tuples: (e_i, e_{i+1}, e_{i+3}), where the product of two ordered members is the third in sequences represented by the circle and six lines of the Fano Plane shown in Fig. 37, so, too, the seven notes and their complements form seven different musical scales. The analogy between the imaginary octonions and the musical intervals is not perfect because multiplication of the former is non-commutative, whereas multiplication of the values of the latter is commutative. However, it is not the imaginary octonions per se that should be compared with musical intervals but, rather, the seven combinations of one, two and three octonions in each 3-tuple:

$$e_i, \ e_{i+1}, \ e_{i+3}$$
$$e_ie_{i+1}, \ e_{i+1}e_{i+3}, \ e_{i+3}e_i \qquad \text{(6 permutations)}$$
$$e_ie_{i+1}e_{i+3} \qquad \text{(6 permutations)}$$

Each 3-tuple has 12 permutations of two or three imaginary octonions. Similarly, as $e_{i+3}e_{i+1} = - e_{i+1}e_{i+3} = - e_i = (-e_{i+3})(-e_{i+1})$ and $e_i^{-1} = -e_i$, then $(e_{i+3})^{-1}(e_{i+1})^{-1} = (e_i)^{-1}$, that is, the inverses of each octonion in a 3-tuple also form a 3-tuple with 12 permutations of pairs or triplets. The seven octonions therefore have (7×12=84) permutations, as do
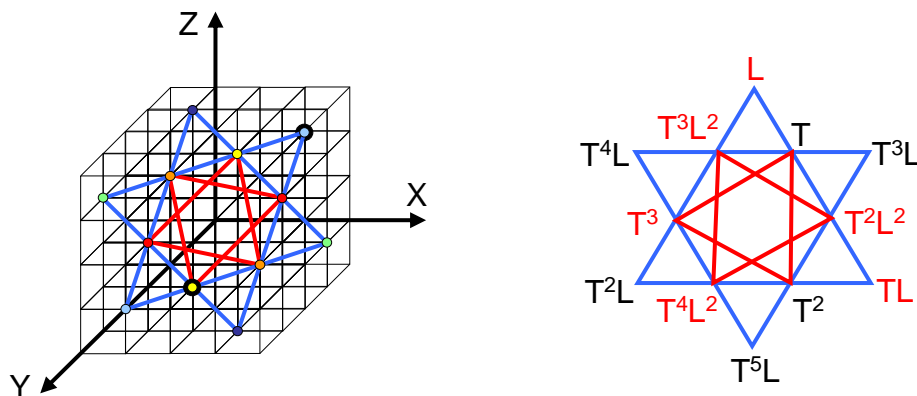

Figure 38. The 12 roots of G_2 (shown as coloured dots) form two nested Stars of David with its two simple roots at vertices of each star (shown as black dots). The 12 notes of the seven musical scales form a similar pattern with opposite star points denoting pair of notes that are complementary.

their seven inverses. Each permutation reduces to one of the octonions. The seven 3-tuples of octonions and the seven 3-tuples of their inverses display (84+84=**168**) distinct permutations of their members. Compare this with the fact[25] that the seven musical scales have 84 rising intervals between their notes that are repetitions of the basic set of Pythagorean intervals, which means that they also have 84 falling intervals, that is, intervals between two notes falling in pitch. The number of rising intervals in each scale that are repetitions of the Pythagorean intervals varies from scale to scale:[26]

C scale	D scale	E scale	F scale	G scale	A scale	B scale
13	12	13	12	11	11	12

Total = 84.

Average = 84/7 = 12.

This is unlike the number of permutations of pairs and triplets of imaginary octonions in each 3-tuple, which is the same (12) in each one. In terms of notes, the composition of the 84 Pythagorean intervals is

D	E	F	G	A
28	11	23	17	5

The interval 243/128 for note B and the interval 2 for the octave are absent because they are the only ones that never appear more than once in a scale (in the former case, the C, E, F & B scales).

The smallest of the five exceptional groups, the rank-2 group G_2 has 14 roots. It has two simple roots $(0,1,-1)$ and $(1,-2,1)$ denoted by black dots in Fig. 38 and 12 roots consisting of six pairs:

red Star of David
$$(1,-1,0) \qquad (-1,1,0)$$
$$(-1,0,1) \qquad (1,0,-1)$$
$$(0,1,-1) \qquad (0,-1,1)$$

blue Star of David
$$(2,-1,-1) \qquad (-2,1,1)$$
$$(-1,2,-1) \qquad (1,-2,1)$$
$$(-1,-1,2) \qquad (1,1,-2)$$

They form two nested Stars of David. The red star shown in Fig. 39 denotes the first triplet of roots and their inversions and the blue star denotes the second triplet of roots and their inversions.

Compare this with the seven pairs of intervals and their complements between notes in the seven musical scales:

$$1 \qquad 2 (= T^5L^2)$$
$$\color{red}{L} \qquad T^5L$$
$$T \qquad \color{red}{T^4L^2}$$
$$\color{red}{LT} \qquad T^4L$$
$$T^2 \qquad \color{red}{T^3L^2}$$
$$LT^2 \qquad T^3L$$
$$\color{red}{L^2T^2} \qquad \color{red}{T^3}$$

The tonic and the octave play the role of the two simple roots and the six pairs of intervals and their complements correspond to the six pairs of roots of G_2 and their inversions. This correspondence permits a nested Star of David representation of the 12 intervals (Fig. 38). Complementarity of intervals corresponds to spatial inversion of roots of G_2. Diametrically opposite vertices of each star signify an interval and its

complement. Whatever the assignment of intervals to star points, the members of one triplet have tone ratios in the same relative proportions as those of the other triplet because the latter are their complements with tone ratios that, being reciprocally related, are in the same proportions, although reversed in order. The two red triplets (T, T^2, T^3) and (T^2L^2, T^3L^2, T^4L^2) have tone ratios in the relative proportions $1:T:T^2$, i.e., those of the perfect 4th, the perfect 5th and the major 6th of the C scale. What is non-
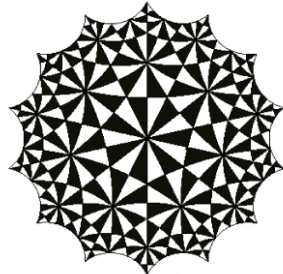

Figure 39. The **168** black triangles of the Klein Configuration represent the **168** automorphisms of the Klein Quartic. The 14 sectors represent the 14 roots of G_2.

trivial, however, is that the two blue triplets (L, LT, LT^2) and (T^3L, T^4L, T^5L) have tone ratios in the *same* proportion. If M-theory is isomorphic to music even more than what previous articles have revealed, this appearance of the same relative proportions between intervals represented by different Stars of David may be telling us that subgroups of rank 3 such as SU(3) play a role that extends beyond that already known
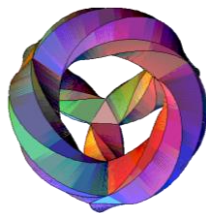

Figure 40. The 7-fold symmetry of the Klein Configuration and the 24-fold symmetry of the tetrahedral 3-torus creates the (7×24=**168**)-fold symmetry of the Klein Quartic.

for it as the gauge symmetry group governing the colour force between quarks. Perhaps the analogy between the root structure of G_2 and the 14 basic notes of the seven musical modes is indicating that there is a fundamental representation of SU(7) with **48** gauge fields consisting of an SU(3) triplet, an SU(3) antitriplet and two SU(3) singlets?

The Klein Configuration is the {7,3} tiling in the hyperbolic plane of the **168** automorphisms of the Riemann surface of the famous 'Klein Quartic':

$$x^3y + y^3z + z^3x = 0.$$

As a surface of genus 3, the Klein curve is a Hurwitz curve having the maximum number of automorphisms for a surface of genus 3. Figure 39 depicts these transformations as black hyperbolic triangles distributed over 14 sectors of the Klein Configuration, 12 per sector, to create 24 heptagons with **168** triangular sectors that cover a 3-torus (Fig. 40). As the **168** element PSL(2,7) of the Klein Configuration is isomorphic to the symmetry group SL(2,3) of the Fano Plane that describes octonion multiplication and as the 14-dimensional G_2 is the group of automorphisms of the octonions, the 12 triangles in each of the sectors of the Klein Configuration represent each of the 14 elements of G_2. These sectors correspond to the 14 notes of the seven musical scales and their **168** triangles correspond to the **168** repeated, rising and falling Pythagorean intervals between these notes (or, alternatively, to the **168** intervals between the notes above the tonic of the eight D–D' scales).

The (2,3) torus knot is a knot that winds twice around a 1-torus as it makes three

revolutions. It can wind on a 3-torus as well. (Fig. 41). The former is the surface around which each whorl of the UPA/heterotic superstring winds 1680 times. The latter is the surface in which the Klein Configuration can be embedded. As the octonions provide a natural basis for the Lie algebra of E_8 and are connected through the Fano Plane to the

(a)

(2,3) torus knot wound on the 1-torus

a whorl of the UPA is a helical coil with 1680 turns wound on a 1-torus.

(b)

(2,3) torus knot wound on the 3-torus

Tiling on the 3-torus of the **168** automorphisms of the Klein Quartic.
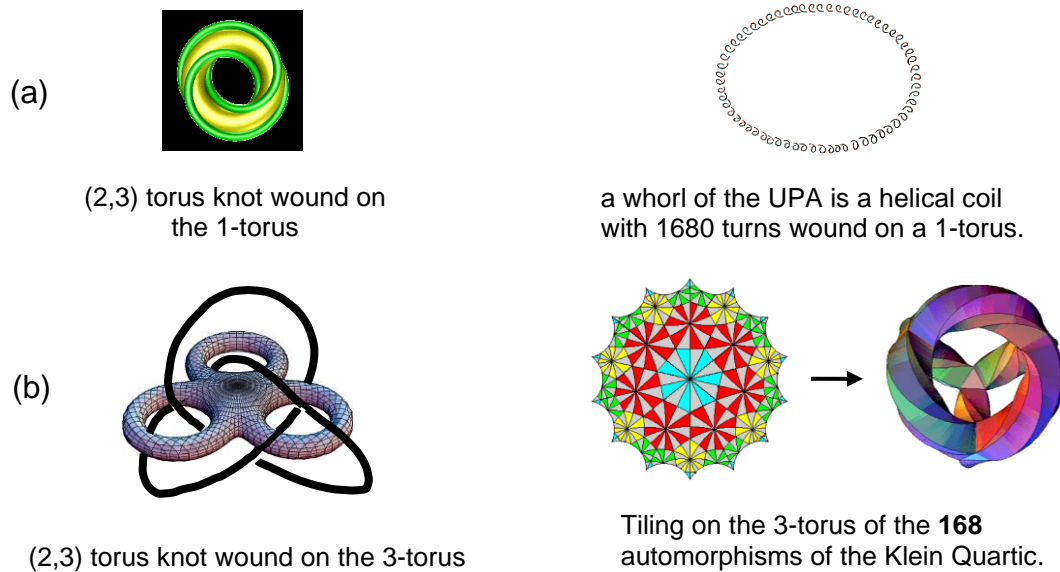
Figure 41. The 1-torus (a) on which the (2,3) torus knot winds is the surface around which each whorl of the heterotic superstring coils 1680 times. The knot can also wind around a 3-torus (b), which is the surface in which can be embedded the Klein Configuration representing the Riemann surface of the Klein Quartic with **168** automorphisms denoted by the triangular sectors of 24 heptagons.

Klein Quartic, the (2,3) torus knot determines *both* the toroidal form *and* the winding number 1680 of the $E_8{\times}E_8$ heterotic superstring.

Because it is well-known to mathematicians that G_2 is the automorphism group of the octonions, perhaps it should not come as a surprise that the pattern of basic intervals in the seven musical scales is similar to the root structure of G_2, given that we have already demonstrated a correspondence between intervals and the properties of octonions that is too rich to be merely coincidental. Nevertheless, it is still remarkable that the same mathematical structure should appear in the 8-component octonions and

Table 6. Number values of the Sephiroth in the four Worlds.

Sephirah	Title	Godname	Archangel	Order of Angels	Mundane Chakra
Kether	620	21	314	833	636
Chokmah	73	15, 26	248	187	140
Binah	67	50	311	282	317
Chesed	72	31	62	428	194
Geburah	216	36	131	630	95
Tiphareth	1081	76	101	140	640
Netzach	148	129	97	1260	64
Hod	15	153	311	112	48
Yesod	80	49, 363	246	272	87
Malkuth	496	65, 155	280	351	168

in the seven 8-note musical scales — even as far as the number of their repeated rising and falling intervals being the order of the symmetry group SL(3,2) for octonions! Moreover, the pattern of intervals of the musical scales is independent of the pitch of a particular octave, remaining true whether the pitch of the tonic is several hundred cycles per second or 10^{24} cycles per second — the frequency associated with subatomic processes, although such fast vibrations would no longer amount to the kind of sound that is detectable to the human ear, which is only sensitive to sounds spanning up to about ten octaves. The remarkable analogy between the musical scales and octonions strongly suggests that, far from being a now defunct musical scale that is of interest only to historians of science and music, the Pythagorean musical scale shares with octonions a vital role in the unified, holistic description of the real world. This is further strongly indicated by the analogous structures of the polyhedral Tree of Life and the ancient musical modes of the Roman Catholic Church. The universality of this system, whose 2-dimensional counterpart is the outer and inner forms of the Tree of Life, means that it has many different layers of interpretation. Music is one such layer, and it provides insight into the general meaning of the polyhedral Tree of Life.

References

[1] Phillips, Stephen M. Article 14: "Why the Greek Musical Modes Are Sacred," http://www.smphillips.mysite.com/article14.pdf, pp. 4–5.

[2] Chappell, W. *The History of Music*, vol. 1 (London, 1874), pp. 50–51.

[3] Ibid, p. 60.

[4] Numbers in boldface are the number values of the Sephiroth, their Godnames, their Archangels and Mundane Chakras, shown in Table 6 above. They are calculated by means of the ancient practice of gematria, wherein a number is assigned to each letter of the alphabet, thereby giving a number value to a word that is the sum of the numbers associated with its letters.

[5] Phillips, Stephen M. Article 16: "The Tone Intervals of the Seven Octave Species and Their Correspondence with Octonion Algebra and Superstrings," http://www.smphillips.mysite.com/article16.pdf, pp. 9–12.

[6] Phillips, Stephen M. Article 4: "The Godnames Prescribe the Inner Tree of Life," http://www.smphillips.mysite.com/article04.pdf, pp. 4–5.

[7] Ref. 5, p. 13.

[8] Ibid. Compare Tables 3–9, pp. 9–11.

[9] Ibid, p. 10.

[10] Ibid, p. 13.

[11] Ibid.

[12] Ibid, pp. 17, 21.

[13] Phillips, Stephen M. Article 29: "The Triakis Tetrahedron and the Disdyakis Triacontahedron Embody the Fine Structure Constant and the Structural Parameter of the Heterotic Superstring," http://www.smphillips.mysite.com/article29.pdf, pp. 4–5.

[14] Phillips, Stephen M. Article 3: "The Sacred Geometry of the Platonic Solids," http://www.smphillips.mysite.com/article03.pdf, p. 10.

[15] Ref. 5, pp. 9-11.

[16] Phillips, Stephen M. Article 27: "How the Disdyakis Triacontahedron Embodies the Structural Parameter 1680 of the $E_8{\times}E_8$ Heterotic Superstring," http://www.smphillips.mysite.com/article27.pdf, pp. 14–16.

[17] Ref. 5, pp. 17, 21.

[18] Phillips, Stephen M. Article 26: "How the Seven Musical Scales Relate to the Disdyakis

Triacontahedron," http://www.smphillips.mysite.com/article26.pdf, pp. 9–10.

[19] Phillips, Stephen M. Article 24: "More Evidence for the Disdyakis Triacontahedron as the 3-dimensional Realisation of the Inner Tree of Life and its Manifestation in the $E_8 \times E_8$ Heterotic Superstring," http://www.smphillips.mysite.com/article24.pdf, pp. 6–7.

[20] Ref. 13, pp. 3, 8.

[21] Phillips, Stephen M. Article 25: "The 33 Vertex Sheets of the Disdyakis Triacontahedron Signify the 33 Tree Levels of Ten Overlapping Trees of Life," http://www.smphillips.mysite.com/article25.pdf, pp. 13-15, 20.

[22] Phillips, Stephen M. Article 12: "New Pythagorean Aspects of Music and Their Connection to Superstrings," http://www.smphillips.mysite.com/article12.pdf, pp. 24–32.

[23] Phillips, Stephen M. Article 22: "The Disdyakis Triacontahedron as the 3-dimensional Counterpart of the Inner Tree of Life," http://www.smphillips.mysite.com/article22.pdf, p. 14.

[24] For proof, see ref. 4 in ref. 15.

[25] Ref. 5, pp. 17–18.

[26] Ibid, p. 21.