

ARTICLE 16

The Tone Intervals of the Seven Octave Species & Their Correspondence with Octonion Algebra and Superstrings

by

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“The first and noblest application of music is in offering the tribute of praise to the immortals; the next is the purifying, regulating and harmonizing the soul.”

Plutarch

Abstract

The likelihood that the ancient Egyptians passed on their knowledge of the seven types of musical scales to Pythagoras, Plato and others who studied with them makes the currently accepted view that the musical modes of ancient Greece were keys rather than octave species highly implausible. The ten Hebrew Godnames known to Kabbalah are found to prescribe the intervallic composition of these scales. Both Plato's Lambda and the first four polygons enfolded in the inner form of the Tree of Life embody the minimum number of different intervals between the notes of the seven modes. Remarkable parallels between the intervallic composition of the latter and the tetrahedral generalisation of the Lambda are demonstrated. There is similar correspondence with the oscillatory form of the superstring. This is because superstring theory is connected to the mathematics of octonions, whose properties are analogous to musical intervals. Remarkable correspondence exists between the 168 permutations of the seven 3-tuples of octonions, the 168 automorphisms of the Klein Quartic, the 168 repetitions of the diatonic intervals of the seven octave species and the superstring structural parameter 168. They indicate that the oscillations performed by superstrings are akin to music played with the notes of the seven musical scales.

I. Ancient Greek musical modes — octave species or keys?

Controversy has persisted to this day over the issue of whether the ancient Greeks classified their different types of music according to keys or to octave species. Did the names Dorian, Phrygian, Lydian, etc. refer to specific keys conferring a distinct moral characteristic, or *ethos*, or were these qualities of music due to their being played in scales that differed in the ordering of what we would now call their tones and semitones? The former view is the more accepted today. For example, Rita Steblin, the world-renowned musical scholar, states “The ancient Greeks

believed that each of their keys, which they identified by the tribal names Dorian, Phrygian, etc., possessed a strongly marked ethical character.”¹ This opening statement in her treatise on the history of the psychological attributes of musical keys may state what is now commonly accepted by many (but not all) scholars but it still begs the question. If the *ethos* of ancient Greek music was really a matter of higher and lower pitch, one has, like Donald Grout, one of the giants of American musicology, to ask: “... how could a mere transposition by one tone up or down account for the difference in *ethos* which Plato postulates between the Dorian and the Phrygian modes?” ... Its “only audible effect would be a redistribution of the tones and semitones within one and the same octave range...”² It is quite possible that the different ethical attributes of the ancient modes had more to do with the nature of the songs and melodies of the different races from whom the ancient Greeks descended. No one, however, knows how the music of ancient Greece sounded. In the absence of reliable, surviving evidence like musical score sheets or playable musical instruments, one can argue either case, citing various authorities to support one’s position, whilst at the same time ignoring past and present writers of equal prestige who take the contrary view. As Donald Jay Grout remarked: “...we do not know, and in all probability we shall never know, exactly what were the modes about which Plato and Aristotle wrote, or how these may have been different from the modes as understood by later writers. ... Each of the various hypotheses about the modes has claims to credibility, and each is warmly defended by its champions. No field of musicology has produced a richer crop of disputation from a thinner soil of fact.”³ This position of neutrality seems a more honest attitude to take than that of Steblin, whose partisanship deceives her readers into thinking that the argument was long won, whereas controversy persists amongst scholars on this issue.

The problem hinges on the question of whether the Greek musical terms *harmonia* and *tonos* meant different things or the same thing. Some scholars, who believe that the ancient Greeks knew only keys argue for the latter, whilst others propose that the Greek modes had a double meaning. For the latter, the use of the word *harmonia* indicated a “mode” and signified the order in which the intervals followed one another, i.e., the species of the octave scale, whilst *tonos* signified a fixed pattern of intervals that could be transposed in pitch, i.e., a key. Unless one is wedded to the implausible view that all the modes — however dissimilar in *ethos* — were the same diatonic scale, differing only by pitch, it is hard to believe that two different musical terms meaning the same would have been in use in the Greek musical lexicon. They are, surely, evidence that the ancient Greeks *did* know of both keys and octave species. The historical evidence cannot decide whether the *ethos* of their modes was due to the former or to the latter. Why, however, should it be a case of one or the other? It is possible that *both* played a role in defining the distinctions between modes and that this ambiguity led to the disagreements between writers as to the correct *ethos* to be attached to a particular mode. They may have been all correct, because the modality ascribed to the same pieces of music might not over time have always been determined by the same criterion! This possibility seems to have been ignored in the longstanding, polarised debate.

Another argument used against the octave species interpretation of Greek modes is that the church modes had Byzantine origins, their four authentic and four plagal modes arising from the Byzantine *octoechoi*, which, too, were grouped into four pairs and which were not scales but collections of melodic motives expressing more or less the same quality of feeling. However, the fact that early, Western musical theorists deduced the scales from which these groups of motives were formed and then incorporated (wrongly, as it turned out) ancient Greek names of modes into their labelling of these scales does not imply that the original Greek modes differed by key. This is a non sequitur, although a convenient one for advocates of the key interpretation of modes to make. Yet it is implicit in the arguments used by Steblin against the interpretation of the Greek modes as octave species. In her discussion of the origins of Christian church music, she is satisfied to stop at the Byzantine era, not bothering to ask whether its music was influenced by that of Greece, Egypt, Syria and other regions where Christians settled. As Grout points out, the

early Christians were Jews and would have borrowed their music from the chants of the Hebrew synagogues, which had melodies based upon modes.⁴ Far from being, as she supposes, a false attribution to the ancient Greeks by early Christian musical theorists, modes as octave species may have been known to the Greek *cognoscenti* and may have influenced the creation of the church modes *indirectly* through the largely unknown impact of Greek music on the Byzantine system. In other words, although the church modes were not derived from the Greek musical modes, they could still have amounted to their rediscovery! This possibility is not considered by Steblin in her defence of the orthodox position in the debate over the meaning of the Greek modes.

The weakness of the scholastic approach to this problem is that it does not discover the truth of the matter but merely recirculates opinions that, however learned, must remain speculative for the obvious reason that these authorities were not contemporaneous with the culture of ancient Greece but, instead, were separated from it by many centuries. No scholar can have read all that was written later about music in that era. Quotations will inevitably be selective and self-serving, excluding whatever contradicts firmly held beliefs. Pressure in the academic world to conform to orthodox opinion can never be overestimated.

2. Ancient evidence for modes being octave species

There are only seven octave species because they are generated by taking successive notes of the seven-note Pythagorean scale as their tonic (it makes no difference to this fact whether musical scales are regarded as ascending or descending):

8.							T	L	T	T	T	L	T	Hypomixolydian (Mode 8, D' scale)
7.							T	T	L	T	T	T	L	Hypolydian (Mode 6, C scale)
6.					L	T	T	L	T	T	T			Hypophrygian (Mode 4, B scale)
5.				T	L	T	T	L	T	T				Hypodorian (Mode 2, A scale)
4.			T	T	L	T	T	L	T					Mixolydian (Mode 7, G scale)
3.		T	T	T	L	T	T	L						Lydian (Mode 5, F scale)
2.	L	T	T	T	L	T	T							Phrygian (Mode 3, E scale)
1.	T	L	T	T	T	L	T							Dorian (Mode 1, D scale)

(T is the Pythagorean whole tone interval of $9/8$, which is slightly sharper than its equal-tempered counterpart, and L is the Pythagorean leimma, an interval of $256/243$, which is somewhat flatter than the semitone). It must be emphasized that the modal labels shown here are those of the church. It is now recognised that the early church musical theorists assigned the Greek labels Dorian, Phrygian, Lydian, etc to successive octaves of the ascending musical scale, not realising that in ancient Greece it was always regarded as descending. This caused mislabelling, so that the *ethos* that the church associated with the modes sometimes disagreed with ancient opinion owing to their misnaming. It was also confusing to number the four Authentic Modes by odd integers and the four Plagal (Hypo) Modes by even integers because this destroyed all sense of their order of generation from successive notes of the diatonic scale, as is shown above by the modal numbers being out of sequence. Perhaps the motivation for this was to make Plato's ideal mode — the Dorian — the first in the list.

Given that the ancient Egyptians had an 8-note scale and that Pythagoras spent many years studying mathematics, music and astronomy with the Egyptian priesthood, it is very likely that he did not discover the mathematics of the musical scale, as popular textbooks on the history of science still inform us, but that he learnt it from his Egyptian teachers. It is therefore not hard to believe that he would have passed on his acquired knowledge to his brotherhood of disciples in Croton, Italy, some of whom then imparted it to their students in ancient Greece itself. Indeed, we know that Plato obtained some of the mathematical secrets of the early Pythagoreans by purchasing a book called *Fragments* from the parents of its author, Philolaus of Tarentum (ca. 480–unknown B.C.E.). Philolaus was educated by Lysis, one of the two disciples of Pythagoras

who escaped the massacre of the Pythagorean community at Croton and who subsequently became the first to write down his master's teachings in his *Fragments*.⁵ A tradition is recorded by Strabo⁶ that Plato spent thirteen years of study at Heliopolis, making it highly plausible that he learnt there the mathematics of music. In his *The History of Music*, W. Chappell went as far as to say that the "entire Greek system was mainly derived from Egypt, Phoenicia, Babylon, or other countries of more ancient civilisation than Greece."⁷ He pointed out that:

*"Diodorus Siculus says that the musicians and poets of Greece visited Egypt for the purpose of improvement, and that the Egyptian priests had records of their visits in their books. The first two names so recorded were those of Orpheus and Musæus, and Homer followed. So, at least, the Egyptians claimed to have taught music and poetry to the Greeks at a very early period. The later names in Diodorus's list, such as those of Solon, Plato, and others, are admittedly authentic."*⁸

Furthermore, he said:

*"Nichomachus, quoting Pythagoras and Plato, tells us that the Egyptians ascribed twenty-eight sounds to the universe, calling it "twenty-eight sounding." So the Egyptians must have had twenty-eight sounds, i.e., twenty-eight notes, in their scales. That is the precise total number of Greek notes, in their greater and lesser perfect systems combined, and including all their scales — Diatonic, Chromatic, and Enharmonic. Neither in Egypt nor in Greece was there an actual limit to twenty-eight sounds, because all scales were transposable, but only twenty-eight notes could be defined, starting from any given pitch. Euclid, Nicomachus, Aristides Quintilianus, and others, enumerate the Greek scales and their notes, and all authors are agreed as to the number being precisely twenty-eight. This most remarkable coincidence between Egypt and Greece seems nevertheless to have escaped the observation of historians of music. If it stood alone, it would almost suffice to prove the origin of Greek music. The number is too peculiar to have been arrived at by accident, within a compass of only two Octaves."*⁹

A simple calculation reveals the significance of this number. Table 1 below shows the tone ratios for each mode, derived from the patterns of intervals that were listed above:

Table 1. Tone ratios of the seven Church modes

Lydian	1	9/8	81/64	729/512	3/2	27/16	243/128	2
Phrygian	1	256/243	32/27	4/3	3/2	128/81	16/9	2
Dorian	1	9/8	32/27	4/3	3/2	27/16	16/9	2
Hypolydian	1	9/8	81/64	4/3	3/2	27/16	243/128	2
Hypophrygian	1	256/243	32/27	4/3	1024/729	128/81	16/9	2
Hypodorian	1	9/8	32/27	4/3	3/2	128/81	16/9	2
Mixolydian	1	9/8	81/64	4/3	3/2	27/16	16/9	2

Turquoise cells contain Pythagorean tone ratios and white cells contain tone ratios not belonging to any octave of the Pythagorean scale. There are 16 such 'non-Pythagorean' notes. The (7×7=49[†]) notes below the octaves of the seven modes comprise (49–16=33) Pythagorean tone ratios, of which seven are 1's, leaving (33–7=26) Pythagorean tone ratios other than 1. *Including the tonic and octave, the seven modes have 28 notes.* This confirms the quotation above. It proves that the ancient Egyptians must have known about seven different octave species, for Pythagoras (556 B.C.E.–unknown) and Plato (428–347 B.C.E.) certainly knew that the Egyptians

[†] Numbers appearing throughout the text in bold type are the number values of the Hebrew Godnames of the ten Sephiroth of the Tree of Life (see previous articles for their tabulated values).

had knowledge of 28 musical sounds. As both studied with the Egyptian priesthood, it is reasonable to infer that they knew about the seven octave species with 28 diatonic notes. This means that the existence of seven musical scales, if not common knowledge, was at least known to musicians who had visited Egypt, to Pythagoras, Plato and to mathematicians that came after him, like Euclid (4th–3rd C B.C.E.), and the Pythagorean, Nicomachus of Geras (about 60–about 120 C.E.). This does not necessarily imply that the musical modes *were* octave species. However, given that the evidence presented here proves that *some* learned, ancient Greeks were, indeed, acquainted with seven scales and that musicians visited Egypt to improve their art, it makes the

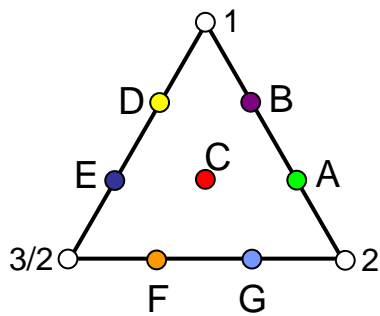
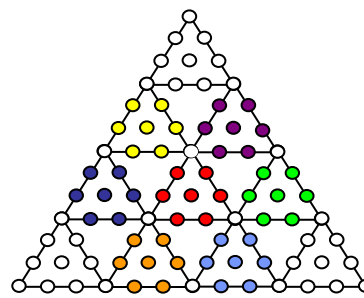


Figure 1. The 1st-order tetractys representation of the seven notes of the Pythagorean musical scale.



- Hypolydian
- Lydian
- Dorian
- Hypodorian
- Mixolydian
- Phrygian
- Hypophrygian

Figure 2. The 2nd-order tetractys representation of the **49** notes of the seven octave species/Greek musical modes.

possibility very likely, if not certain. Scholars who deny this identity are ignoring the fact that music has never recognised borders. It has always been influenced and enriched by styles and talent imported from other countries. It would explain why two ancient Greek musical terms — *harmonia* and *tonos* — were in use, which advocates of the ‘key’ interpretation of modes have, implausibly, to assume to have been synonymous. It seems that the agreement between ancient Egyptians and Greeks as to the number of musical notes and its implication that *both* knew about the existence of seven octave species have escaped the attention of modern musical scholars as much now as it did when Chappell wrote his book in 1874! Otherwise, they would not keep dogmatically denying that musicians of ancient Greece were unacquainted with them. It does not make sense that they knew about different scales yet played only one of them in different keys!

The possibility of seven octave species would have been in accord with Pythagorean principles for the following reason: the Pythagorean tetractys* symbolises the three archetypal and seven formative principles inherent in all holistic systems. The former — the noumenal aspect of a holistic system, are symbolised by the three points or yods at its corners, which give the tetractys its bare shape as a triangle; the latter — its phenomenal nature — are symbolised by the seven yods arranged at the centre and corners of a hexagon (so-called “hexagonal yods”). Just as the first seven notes of an 8-note musical scale are symbolised by the seven hexagonal yods (shown in Figure 1 as coloured dots, so the **49** notes below the octaves of the seven octave species are symbolised by the analogous **49** hexagonal yods of the next higher-order tetractys — the 2nd-order tetractys (Fig. 2). The seven octaves species represent a more differentiated version of the *whole* system of musical notes. They constitute *another whole*. The

* The author calls this the ‘1st-order tetractys,’ the point symbolising the Pythagorean Monad being the 0th-order tetractys. In general, the nth-order tetractys has 7ⁿ hexagonal yods.

The 4x7 pattern of the 28 intervals of the Pythagorean scale. Arrows joining corners of the octagon denote intervals (arrowheads point towards rising pitch). Red arrows are associated with one square, blue arrows with the other square. Thick lines denote the five tone intervals (9/8) and the two leimmas (256/243). Non-diatonic tone ratios are in red.

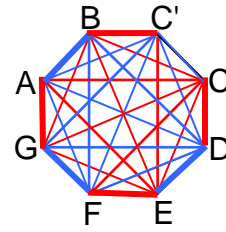
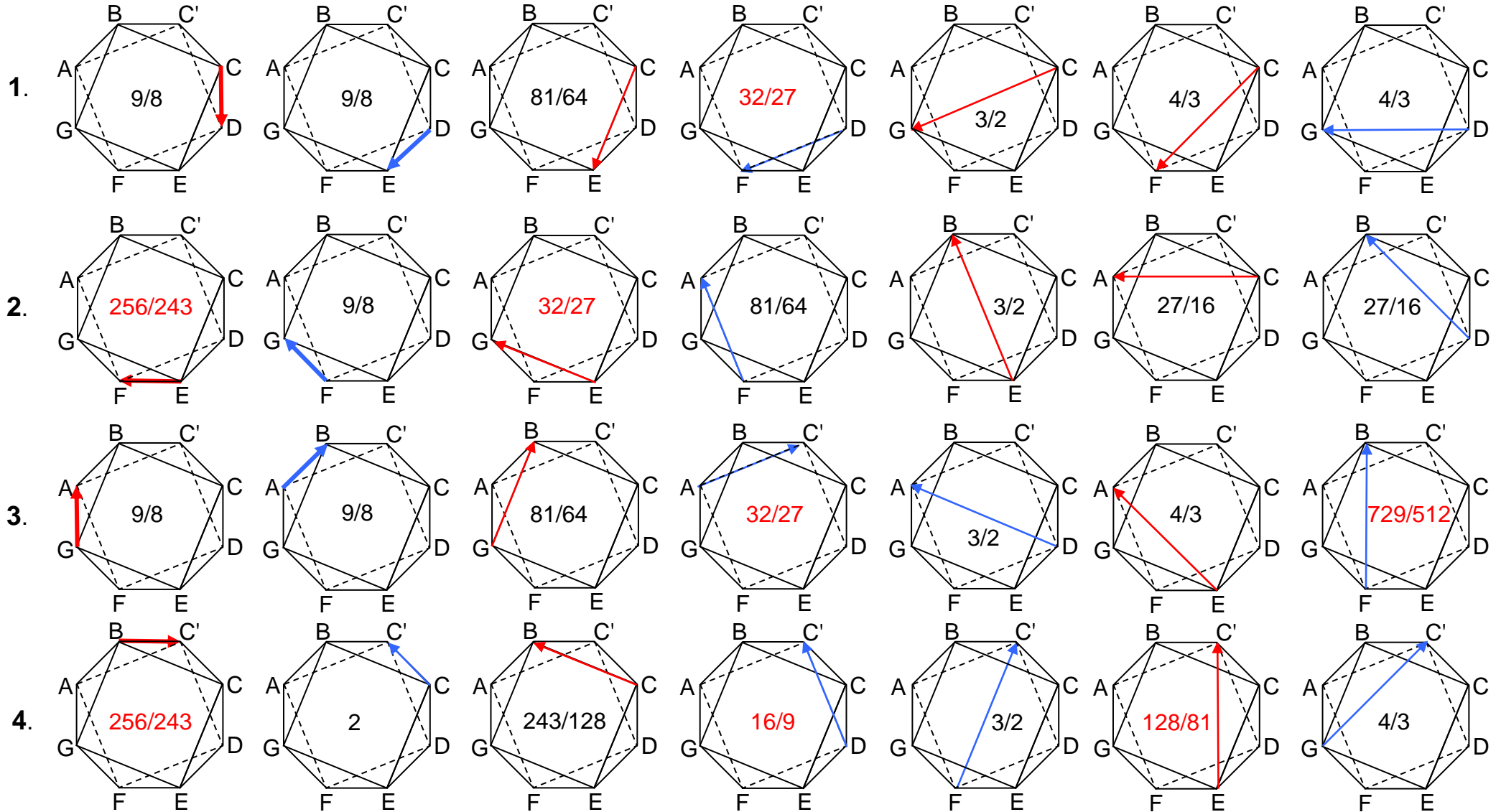


Figure 4

Start with corners C and D of the two squares and trace sequences of four analogous types of lines down each of the seven columns.



yods at the corners of the 1st-order tetractys mark out the shape of this template, from which the other seven yods emerge as *independent, phenomenal qualities of the holistic system*. In the same way, the number 1 denoting the tonic, the number 2 denoting the octave and the number 3/2 denoting the perfect fifth should be assigned to these three yods because 1 and 2 define the pitch range of the musical scale, whilst their arithmetic mean: $(1+2)/2 = 3/2$ is the pitch of the

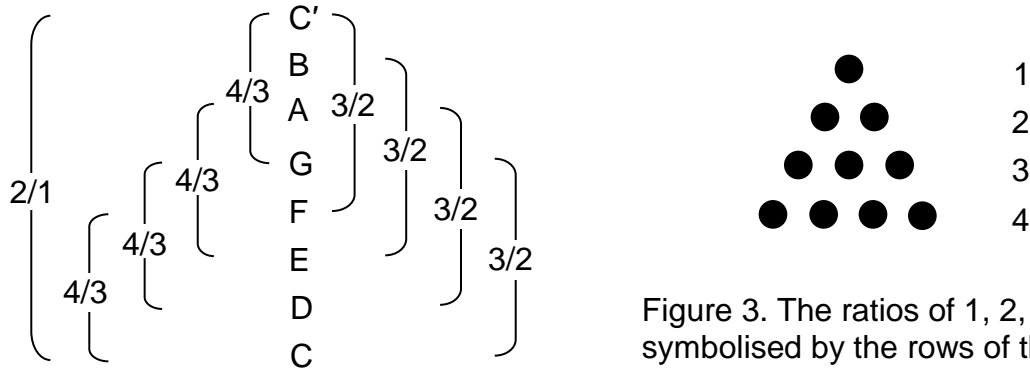


Figure 3. The ratios of 1, 2, 3 & 4 symbolised by the rows of the tetractys are the intervals between notes of the Pythagorean scale.

perfect fifth G from which the other notes of the scale can be generated by jumps of intervals of a perfect fifth or a perfect fourth — the interval between G and the octave C' (Fig. 3).

As the Pythagorean scale has eight notes, they form ${}^8C_2 = 28^*$ pairs or intervals. There are as many intervals in the Pythagorean scale as there are notes in the seven octave species. Representing the eight notes of the scale by the corners of an octagon, these intervals correspond to the straight lines that join pairs of corners (Fig. 4). An octagon is formed by the corners of two squares rotated relative to each other by 45°. By joining corners of each square and grouping analogous connections, the 28 lines group naturally into four sets of seven. Column 1 indicates consecutive edges of one square (shown by solid lines); column 2 shows consecutive edges of the second square (shown by dotted lines); column 3 shows opposite diagonals of each square; column 4 shows corners of one square joined along an edge of the octagon to a corner of the other square; column 5 indicates corners of the second square joined along an edge of the octagon to a corner of the first square; column 6 exhibits corners of the first square joined internally to corners of the other square; column 7 shows corners of the second square joined internally to corners of the first square. The corresponding intervals are indicated inside each octagon. The 20 Pythagorean intervals include five tone intervals of 9/8. The eight non-Pythagorean intervals include two leimmas of 256/243. There are five seconds, three thirds, four perfect fourths, four perfect fifths, two sixths, one seventh and one octave. No one row contains all the seven Pythagorean one ratios. If we do not join corners in a clockwise sequence in each column, at least one row (the top) can contain all seven diatonic notes. It is unimportant which column or row is treated as the first, although it seems appropriate to start (as shown in Figure 4) with the interval from C to D. All that matters in any rearrangement is that similar types of lines remain in the same column.

The following passage from Chappell's book *seems* to give significance to this 4x7 pattern of intervals:

“The Greeks copied the Egyptians in associating musical sounds with the heavenly bodies; but, as they made their computations of time by the lunar month, they connected the twenty-eight notes of the scales with the twenty-eight days of the moon; and the fifteen notes of the Diatonic scale were the fifteen days of the

* The number of combinations of r objects from a set of n objects is ${}^nC_r = n!/(r!(n-r)!)$, where $n! = 1 \times 2 \times 3 \times \dots \times n$.

moon's increase.¹⁰ The Egyptians subdivided the lunar month into weeks through dedicating the first hour of each of the seven days to the seven planets, as the seven deities, who were supposed to watch over them. This association seems to have originated in Babylon. The seven planets and the seven days coincided with the seven notes of the Octave."¹¹

Quintilianus's reference to the **15** notes of the diatonic scale must be an error because the seven octave species have (including the tonic) 14 different notes, of which eight are diatonic, whilst, although the first two octaves of the diatonic scale have **15** notes, eight of them belong to the diatonic scale on the next higher octave and so the seven notes above the first octave do not belong to the 28 notes making up the seven scales, which all belong to the first octave. The numbers of notes of each type are: 1 (x1), 9/8 (x5), 81/64 (x3), 4/3 (x6), 3/2 (x6), 27/16 (x4), 243/128 (x2) & 2 (x1). Suppose that Quintilianus was confusing the 28 notes of the seven octave species with the 28 intervals of the diatonic scale, among which are the eight notes and their seven intervals, totalling **15**, the number that he mentions. Figure 4 shows that, by interchanging columns 1 and 2 and then rows 3 and 4, the first octave with tone ratio 2 will go into the **15th** member of the set of 28 intervals. However, this will not work because, among the **15** notes and intervals of the diatonic scale, is the non-diatonic interval 256/243, a number which is absent from the 28 diatonic tone ratios. Given this, does the set of 28 notes in the seven scales still have a subset made up of **15** notes defined in some way? Fortunately, the list of numbers of notes given above provides a natural solution because the number of notes in the list up the perfect fourth is **15!** Intuitively, this seems right because, just as the **15th** note is the midpoint of the sequence of 28 notes, so the perfect fourth is the middle note of the scale:

1	243/12
9/8	27/16
81/64	3/2
4/3	

As a week corresponds to a row and as there are as many as six notes of the same type, at least one week must have two days with the same note. This is to be expected because, whether they are intervals of one scale or notes of seven scales, the set of 28 musical elements are not all different. However, the quotation above definitely refers to the "twenty-eight notes of the scales," that is, the ancient Egyptians associated the 28 notes of the seven octave species, not the 28 intervals of the Pythagorean scale, with the 28 days of the lunar month. It is the subset of 28 numbers in the turquoise cells of Table 1 that needs to be grouped into four sets of seven. Quintilianus was right about the Egyptians associating lunar days with notes but erred in thinking

Table 2. Diatonic Notes corresponding to the 28 Days of the Egyptian Lunar Month

Week 1	1	9/8	81/64	4/3	3/2	27/16	2	☽
Week 2	9/8	4/3	3/2	4/3	9/8	81/64	4/3	○
Week 3	3/2	27/16	243/128	9/8	4/3	3/2	27/16	☾
Week 4	4/3	3/2	9/8	81/64	3/2	27/16	243/128	●

that the first **15** days were associated with the notes of the first two octaves, some of which do not belong to the notes of the seven scales and so cannot be included with them. Some criterion must determine the ordering of notes into subsets of **15** and 13, as the former can be neither the **15** notes of two octaves nor the **15** notes and intervals of one scale. It seems reasonable to assume that the progression of the days of the lunar month follows the order of the notes in the natural mathematical sequence of the seven modes as they are generated from successive notes

of the Pythagorean scale. There are two possibilities: Mixolydian–Lydian, if the notes are ascending, and Lydian–Mixolydian, if they are descending. Considering the former first, the fourth note of the fourth mode (the Hypolydian), when it is part of the modal sequence of 28 notes, is a perfect fifth, $3/2$ (see Table 1), and it is the **15th** note, which make it fitting that the turning point of the lunar cycle as the full moon begins to wane should coincide with the very musical centre of gravity of the seven scales, namely, the fourth note of the fourth scale in the unfolding series of seven octave species. This is shown in Table 2 as the white cell. Considering the second possibility, according to Table 1, the **15th** note in the set of 28 diatonic notes is E, the third note of the fourth mode, the Hypolydian, with tone ratio $81/64$. This is not the middle of the Hypolydian mode. If the days of the lunar month start with the notes of the Lydian mode, its turning point is not the middle of the middle scale. As symmetry in the modal spectrum should have its counterpart in the lunar cycle, with the two halves of the cycle having their counterparts in the seven scales, the correct sequence of modes is Mixolydian–Lydian. This is indicated in Table 2 by the tone ratios being written in the seven colours of the spectrum. As both the perfect fourth and the perfect fifth occur six times, it is inevitable that some notes appear more than once in a week, however they are assigned to the days.

3. Intervallic structure of the seven modes

The pattern of notes and intervals for each mode are displayed below, together with their numbers of Pythagorean/non-Pythagorean intervals and the minimal numbers of different intervals.

Table 3. Hypolydian Mode

	C	D	E	F	G	A	B	C'
C	1	9/8	81/64	4/3	3/2	27/16	243/128	2
D	8/9	1	9/8	32/27	4/3	3/2	27/16	16/9
E	64/81	8/9	1	256/243	32/27	4/3	3/2	128/81
F	3/4	27/32	243/256	1	9/8	81/64	729/512	3/2
G	2/3	3/4	27/32	8/9	1	9/8	81/64	4/3
A	16/27	2/3	3/4	64/81	8/9	1	9/8	32/27
B	128/243	16/27	2/3	512/729	64/81	8/9	1	256/243
C'	1/2	9/16	81/128	2/3	3/4	27/32	243/256	1

Blue cells contain rising intervals belonging to the basic set of 13 intervals/notes. Violet cells contain their falling intervals. White cells contain repeated intervals.

Number of intervals = 28 (Pythagorean) + 8 (non-Pythagorean) = **36**.

Minimum number of different intervals = 7 (Pythagorean) + 5 (non-Pythagorean) = 12.

Table 4. Mixolydian Mode

	C	D	E	F	G	A	B	C'
C	1	9/8	81/64	4/3	3/2	27/16	16/9	2
D	8/9	1	9/8	32/27	4/3	3/2	128/81	16/9
E	64/81	8/9	1	256/243	32/27	4/3	1024/729	128/81
F	3/4	27/32	243/256	1	9/8	81/64	4/3	3/2
G	2/3	3/4	27/32	8/9	1	9/8	32/27	4/3
A	16/27	2/3	3/4	64/81	8/9	1	256/243	32/27
B	9/16	81/128	729/1024	3/4	27/32	243/256	1	9/8
C'	1/2	9/16	81/128	2/3	3/4	27/32	8/9	1

Number of intervals = 25 (Pythagorean) + 11 (non-Pythagorean) = **36**.
 Minimum number of different intervals = 6 (Pythagorean) + 5 (non-Pythagorean) = 11.

Table 5. Lydian mode

	C	D	E	F	G	A	B	C'
C	1	9/8	81/64	729/512	3/2	27/16	243/128	2
D	8/9	1	9/8	81/64	4/3	3/2	27/16	16/9
E	64/81	8/9	1	9/8	32/27	4/3	3/2	128/81
F	512/729	64/81	8/9	1	256/243	32/27	4/3	1024/729
G	2/3	3/4	27/32	243/256	1	9/8	81/64	4/3
A	16/27	2/3	3/4	27/32	8/9	1	9/8	32/27
B	128/243	16/27	2/3	3/4	64/81	8/9	1	256/243
C'	1/2	9/16	81/128	729/1024	3/4	27/32	243/256	1

Number of intervals = 27 (Pythagorean) + 9 (non-Pythagorean) = **36**.
 Minimum number of different intervals = 7 (Pythagorean) + 6 (non-Pythagorean) = 13.

Table 6. Dorian Mode

	C	D	E	F	G	A	B	C'
C	1	9/8	32/27	4/3	3/2	27/16	16/9	2
D	8/9	1	256/243	32/27	4/3	3/2	128/81	16/9
E	27/32	243/256	1	9/8	81/64	729/512	3/2	27/16
F	3/4	27/32	8/9	1	9/8	81/64	4/3	3/2
G	2/3	3/4	64/81	8/9	1	9/8	32/27	4/3
A	16/27	2/3	512/729	64/81	8/9	1	256/243	32/27
B	9/16	81/128	2/3	3/4	27/32	243/256	1	9/8
C'	1/2	9/16	16/27	2/3	3/4	27/32	8/9	1

Number of intervals = **26** (Pythagorean) + 10 (non-Pythagorean) = **36**.
 Minimum number of different intervals = 6 (Pythagorean) + 5 (non-Pythagorean) = 11.

Table 7. Hypodorian Mode

	C	D	E	F	G	A	B	C'
C	1	9/8	32/27	4/3	3/2	128/81	16/9	2
D	8/9	1	256/243	32/27	4/3	1024/729	128/81	16/9
E	27/32	243/256	1	9/8	81/64	4/3	3/2	27/16
F	3/4	27/32	8/9	1	9/8	32/27	4/3	3/2
G	2/3	3/4	64/81	8/9	1	256/243	32/27	4/3
A	81/128	729/1024	3/4	27/32	243/256	1	9/8	81/64
B	9/16	81/128	2/3	3/4	27/32	8/9	1	9/8
C'	1/2	9/16	16/27	2/3	3/4	64/81	8/9	1

Number of intervals = 25 (Pythagorean) + 11 (non-Pythagorean) = **36**.
 Minimum number of different intervals = 6 (Pythagorean) + 5 (non-Pythagorean) = 11.

Table 8. Phrygian mode

	C	D	E	F	G	A	B	C'
C	1	256/243	32/27	4/3	3/2	128/81	16/9	2
D	243/256	1	9/8	81/64	729/512	3/2	27/16	243/128
E	27/32	8/9	1	9/8	81/64	4/3	3/2	27/16
F	3/4	64/81	8/9	1	9/8	32/27	4/3	3/2
G	2/3	512/729	64/81	8/9	1	256/243	32/27	4/3
A	81/128	2/3	3/4	27/32	243/256	1	9/8	81/64
B	9/16	16/27	2/3	3/4	27/32	8/9	1	9/8
C'	1/2	128/243	16/27	2/3	3/4	64/81	8/9	1

Number of intervals = 28 (Pythagorean) + 8 (non-Pythagorean) = **36**.
 Minimum number of different intervals = 7 (Pythagorean) + 5 (non-Pythagorean) = 12.

Table 9. Hypophrygian Mode

	C	D	E	F	G	A	B	C'
C	1	256/243	32/27	4/3	1024/729	128/81	16/9	2
D	243/256	1	9/8	81/64	4/3	3/2	27/16	243/128
E	27/32	8/9	1	9/8	32/27	4/3	3/2	27/16
F	3/4	64/81	8/9	1	256/243	32/27	4/3	3/2
G	729/1024	3/4	27/32	243/256	1	9/8	81/64	729/512
A	81/128	2/3	3/4	27/32	8/9	1	9/8	81/64
B	9/16	16/27	2/3	3/4	64/81	8/9	1	9/8
C'	1/2	128/243	16/27	2/3	512/729	64/81	8/9	1

Number of intervals = 27 (Pythagorean) + 9 (non-Pythagorean) = **36**.
 Minimum number of different intervals = 7 (Pythagorean) + 6 (non-Pythagorean) = 13.

INTERVALLIC COMPOSITION OF SEVEN MODES

Total number of intervals = $7 \times 36 = 252$.

Total number of intervals other than 1 = $252 - 7 \times 8 = 196$.

Total number of inverse intervals = $7 \times 28 = 196$.

Total number of intervals and their inverses = $252 + 196 = 448$.

Total number of intervals and their inverses other than 1 = $2 \times 196 = 392$.

Total number of Pythagorean intervals = 186 (186 – 56 = 130 Pythagorean intervals other than 1). Total number of Pythagorean intervals and their inverses = $2 \times 130 + 7 \times 8 = 316$.

Total number of Pythagorean intervals and their inverses other than 1 = $2 \times 130 = 260$.

Total, minimum number of different intervals = 46 (Pythagorean) + 37 (non-Pythagorean) = 83

All the pitches of these intervals are pitches of tones of the seven Church modes. Including the seven tonics, the total, minimum number of Pythagorean intervals = $7 + 46 = 53$. Total, minimum

number of different intervals (including that between each note and itself) = 53 (Pythagorean) + 37 (non-Pythagorean) = 90.

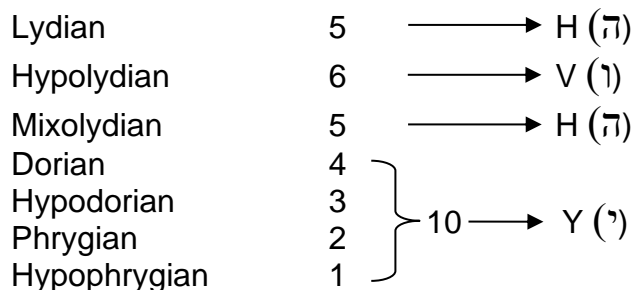
Table 10. Tone ratios of the notes of the seven Church musical modes

Lydian	1	9/8	81/64	729/512	3/2	27/16	243/128	2
Phrygian	1	256/243	32/27	4/3	3/2	128/81	16/9	2
Dorian	1	9/8	32/27	4/3	3/2	27/16	16/9	2
Hypolydian	1	9/8	81/64	4/3	3/2	27/16	243/128	2
Hypophrygian	1	256/243	32/27	4/3	1024/729	128/81	16/9	2
Hypodorian	1	9/8	32/27	4/3	3/2	128/81	16/9	2
Mixolydian	1	9/8	81/64	4/3	3/2	27/16	16/9	2

Table 10 shows the tone ratios of the notes of the seven octave species. White cells indicate non-Pythagorean tone ratios. The modes have notes with thirteen different tone ratios other than 1 (the “basic set”), as well as their thirteen inversions, totalling **26** (27, including 1):*

256/243 9/8 32/27 81/64 4/3 1024/729 729/512 3/2 128/81 27/16 16/9 243/128 2
 243/256 8/9 27/32 64/81 3/4 729/1024 512/729 2/3 81/128 16/27 9/16 128/243 1/2

The number value **26** of *Yahweh* (יהוה), Godname of Chokmah, is the number of *different* rising and falling tone intervals in the seven modes. The Godname prescribes their tonal range. This is consistent with the location of Chokmah in the Tree of Life at the head of the Pillar of Mercy, for this Sephirah denotes the creative power of God, and *Yahweh* lays down the fullness or scope of the archetypal ideas of the Divine Mind as abstract possibility. The number **26** is also the number of diatonic tone ratios other than 1 in the seven modes:



The letter values of YHVVH denote the numbers found in individual modes: Y = 10 denotes the number of Pythagorean notes other than 1 in the Dorian, Hypodorian, Phrygian and Hypophrygian modes, H = 5 denotes the number of such notes in the Mixolydian mode and in the Lydian mode and V = 6 denotes the number of such notes in the Hypolydian mode.

The Godname Yah (יה) with number value **15** prescribes the eight notes of each mode and their seven intervals. This is consistent with its Sephirah Chokmah for the same reason as that given above for *Yahweh*, the later, full version of this Godname. The Godname *Eloha* with number value **36** prescribes the **36** notes and intervals of an 8-note musical scale. The Godname *Ehyeh* with number value **21** prescribes the seven notes below the octave because they have **21** intervals. Of the 56 separate notes in the seven modes, 14 are different notes, leaving (56–14=42) repetitions, where 42 is the **21st** even integer. *Ehyeh* therefore also prescribes those notes that are repeated. Six of them are the tonic 1, leaving (42–6=36) repeated notes above the tonic. *Eloha* prescribes those notes of the seven modes above the tonic that are repeated.

The numbers of intervals of each type in the seven octave species are:

* The rising intervals are listed in ascending order of pitch.

Interval	Number		
1	56		
9/8	35	}	130 Pythagorean intervals
81/64	18		
4/3	30		
3/2	24		
27/16	12		
243/128	4		
2	7		
729/512	5	}	66 non-Pythagorean intervals
256/243	14		
32/27	24		
128/81	9		
16/9	10		
1024/729	4		

There are 130 Pythagorean intervals other than 1 and 130 of their inversions, totalling 260

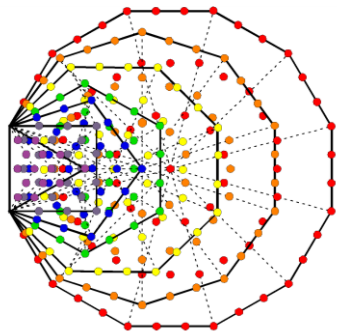


Figure 2. The 260 yods in the seven enfolded polygons outside their shared root edge denote the 260 Pythagorean intervals in the seven Greek modes and their inversions.

(= 26×10). *Yahweh* with number value **26** prescribes the number of Pythagorean intervals and their inversions. This intervallic composition is encoded in the inner form of the Tree of Life as the 260 yods in the seven enfolded polygons outside their shared root edge (Fig. 2). It demonstrates

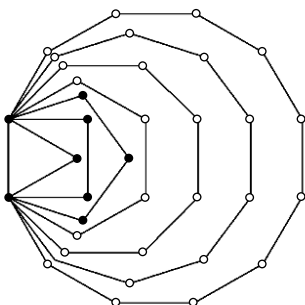


Figure 3. The inner Tree of Life design of an octave species. The 8 (●) corners of the triangle, square and pentagon correspond to its 8 notes and the 28 (○) corners correspond to its other 28 intervals. The two ends of the shared root edge correspond to the tonic C and the octave C'.

the fact that the seven Greek modes constitute an *archetypal musical whole*. The **36** intervals of each mode are symbolised by the **36** corners of these polygons (Fig. 3).

130 is the **129**th integer after 1. The number value **129** of *Yahweh Sabaoth*, Godname of Netzach, is the number of rising, diatonic intervals other than 1 in the seven octave species.

Including the intervals 1, the total number of diatonic intervals and their inversions = $260 + 56 = 316$. The number of diatonic intervals and their inversions other than $\frac{1}{2}$, 1 and 2 = $316 - 7(8+1+1) = 316 - 70 = 246$. This is the number value of *Gabriel*, Archangel of Yesod. It is the number of true intervals and their inversions in the seven modes other than whole octaves.

There are 66 non-diatonic intervals and 66 of their inversions, totalling 132. 66 is the **65**th integer after 1, where **65** is the number value of *Adonai*, Godname of Malkuth. This also prescribes the

number of diatonic intervals other than 1 because 130 is the **65th** even integer. The number of intervals other than 1 = $130 + 66 = 196$.

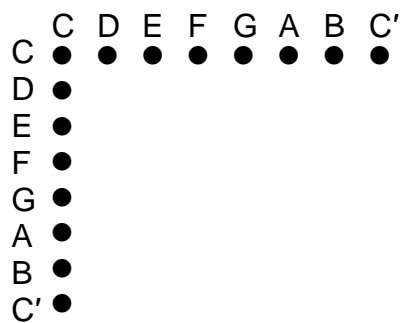
The number of intervals (including 1) = $56 + 196 = 252$. The number of intervals (including 1) and their inversions = $56 + 2 \times 196 = 448$. As a check, this is $7 \times 8 \times 8$, the number of entries in the seven 8×8 arrays of intervals.

Table 11 indicates that the Pythagorean scale has 13 repetitions of diatonic notes and three repetitions of non-diatonic intervals (purple cells), totalling 16, the **15th** integer after 1. This

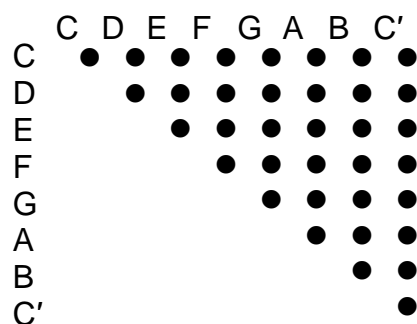
Table 11

	C	D	E	F	G	A	B	C'
C	1	9/8	81/64	4/3	3/2	27/16	243/128	2
D		1	9/8	32/27	4/3	3/2	27/16	16/9
E			1	256/243	32/27	4/3	3/2	128/81
F				1	9/8	81/64	729/512	3/2
G					1	9/8	81/64	4/3
A						1	9/8	32/27
B							1	256/243
C'								1

shows how the Godname *Yah* with number value **15** prescribes this pattern of intervals of the Pythagorean scale. It more fundamentally prescribes the 8×8 array of intervals (including 1) and their inverses in terms of the **15** entries forming two adjacent sides of this square array:



The number value **36** of *Eloha*, Godname of Geburah, is the number of intervals between the eight notes of any scale, counting the intervals 1 between the notes themselves:



The number value **49** of the Godname *El Chai* is the number of intervals between successive notes of the seven octave species:

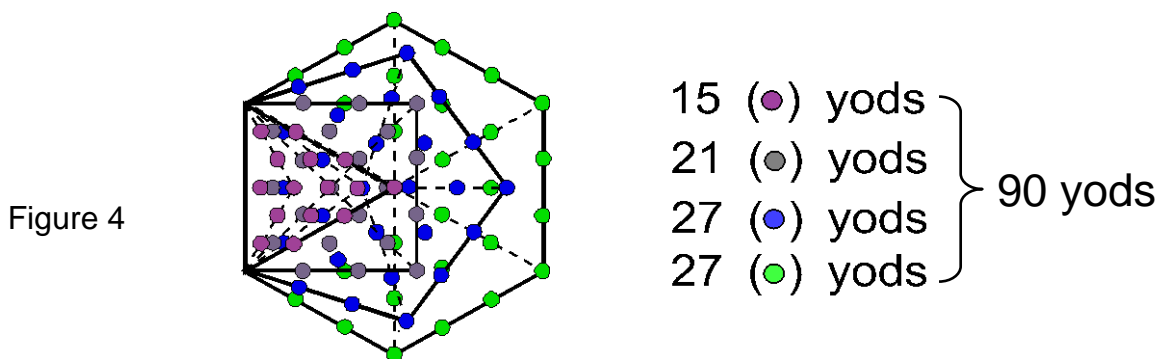
	C	D	E	F	G	A	B	C'
Mixolydian	●	●	●	●	●	●	●	●
Phrygian	●	●	●	●	●	●	●	●
Lydian	●	●	●	●	●	●	●	●
Dorian	●	●	●	●	●	●	●	●
Hypophrygian	●	●	●	●	●	●	●	●
Hypolydian	●	●	●	●	●	●	●	●
Hypodorian	●	●	●	●	●	●	●	●

The number of different intervals between the 56 notes of the seven modes is 90. Remarkably, this is the sum of the integers in the tetractys formed by Plato's Lambda:

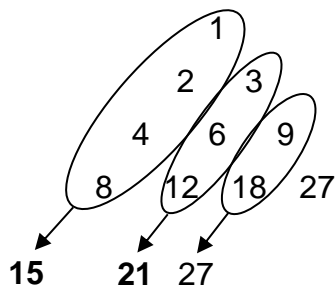
1
 2 3
 4 6 9
 8 12 18 27.

27, the largest integer in the Lambda, is the number of different tone intervals and their inversions between the notes of the seven modes. The sum of the integers 1, 8 and 27 at the corners of the tetractys is **36**, which is the number of intervals in each mode.

The four simplest polygons in the inner form of the Tree of Life have 90 yods outside their shared



root edge (Fig. 4). This is a remarkable illustration of the Tetrad Principle defining a set of geometrical objects that embody a number of universal significance — in this case the total minimum number of different intervals between the notes of the seven musical modes. As confirmation that this is not just coincidence, the numbers of yods in the triangle, square, pentagon and hexagon outside their root edge are, respectively, **15, 21, 27** and **27**, which are the sums of the integers along diagonal rows of the Lambda tetractys:



This beautiful parallel between the inner geometry of the Tree of Life and the arithmetic properties of the Lambda serves to demonstrate their archetypal character.

The minimal set of 90 different intervals in the seven modes includes ones where an interval occurs once in a mode, e.g. the interval 1024/729 in the Lydian mode. The basic set has 14 different ones, so (90–14=76) intervals are repeated. The number value **76** of *Yahweh Elohim*, Godname of Tiphareth, is the minimum number of these repeated intervals.

Inspection of the tables of intervals reveals that, of the 66 non-diatonic intervals, 16 (=4²) are between the tonic C and other notes. There are (66–16=50) non-diatonic intervals between notes other than C. This shows how the Godname *Elohim* with number value **50** prescribes non-diatonic intervals between notes other than the tonic. The number of non-diatonic intervals and their inverses between the notes and the tonic C or octave C' is 32, which is the **31st** integer after 1. This shows how the Godname *El* assigned to Chesed with number value **31** prescribes the non-diatonic intervals between notes and opposite ends of each mode.

Similarly, the tables show that, of the 130 diatonic intervals, 33 (= 1! + 2! + 3! + 4!) intervals are between C and other notes, leaving (130–33=97) diatonic intervals between notes other than the tonic. 97 is the **49th** odd integer, showing how the Godname *El Chai* assigned to Yesod with number value **49** prescribes this number. 97 is also the number value of *Haniel*, the Archangel of Netzach. Including the unit intervals between the 56 notes themselves, the number of diatonic intervals between notes other than the tonic = 97 + 56 = **153**. This is the number value of *Elohim Sabaoth*, the Godname assigned to Hod. The number of diatonic and non-diatonic intervals between C and other notes = 33 + 16 = **49**. Therefore, *El Chai* prescribes the intervals in the seven modes between the tonic and other notes, as well as the diatonic intervals between notes above the tonic.

Below is a summary of how the Godnames prescribe properties of the seven octave species.

Kether: 21.	21 intervals between 7 notes below the octave; 42 repetitions of basic set of 14 notes forming the 7 modes, where 42 = 21st even integer.
Chokmah: 15.	8 notes in each mode separated by 7 intervals, totalling 15 notes & intervals.
26.	number of different intervals other than 1 and their inverses; number of harmonics in 7 cycles of 7 modes.
	260 (=26×10) diatonic intervals and their inverses.
Binah: 50.	number of non-diatonic intervals between notes above the tonic.
Chesed: 31.	number of non-diatonic intervals and their inverses between notes and the tonic or octave = 32, which is the 31st integer after 1.
Geburah: 36.	number of repetitions of basic set of 14 notes above the tonics of the 7 modes; each mode has 8 notes and 28 intervals, totalling 36 notes and intervals.
Tiphareth: 76.	minimum number of repeated intervals.
Netzach: 129.	130 diatonic intervals, where 130 = 129th integer after 1.
Hod: 153.	number of diatonic intervals (including those between the notes themselves) between notes above the tonic.
Yesod: 49.	number of intervals between successive notes of 7 modes; number of diatonic and non-diatonic intervals between the tonic and other notes; 97 diatonic intervals between notes other than the tonic, where 97 = 49th odd integer.
Malkuth: 65.	number of non-diatonic intervals = 66, where 66 = 65th integer after 1; 130 diatonic intervals, where 130 = 65th even integer.

4. The Lambda Tetrahedron

The number of diatonic and non-diatonic intervals between notes other than the tonic = $97 + 50 = 147$. Similarly, these have 147 inverse intervals on the opposite sides of the diagonals of Tables

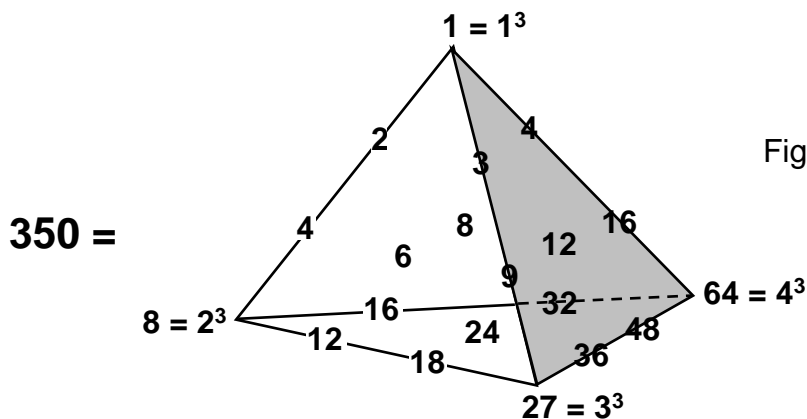


Figure 5. The Lambda Tetrahedron.

3–9. Therefore, there are $(147+147=294)$ diatonic and non-diatonic intervals between notes other than C together with their inverses. Including the 56 intervals 1 on the diagonals, there are $(56+294=350)$ such intervals. They comprise **50** non-diatonic intervals and their **50** inverses, that is, 100 non-diatonic intervals, leaving $(350-100=250)$ diagonal intervals, diatonic intervals between notes other than the tonic and their inverses (check: $250 = 56 + 97 + 97$). The division:

$$350 = 100 + 250$$

between the 100 non-diatonic intervals and their inverses and the 250 diatonic ones and their inverses has its parallel in the tetrahedral version of the Lambda tetractys discussed in Article 11.¹² The 20 integers arranged in tetractyses on the four faces of the tetrahedron add up to 350

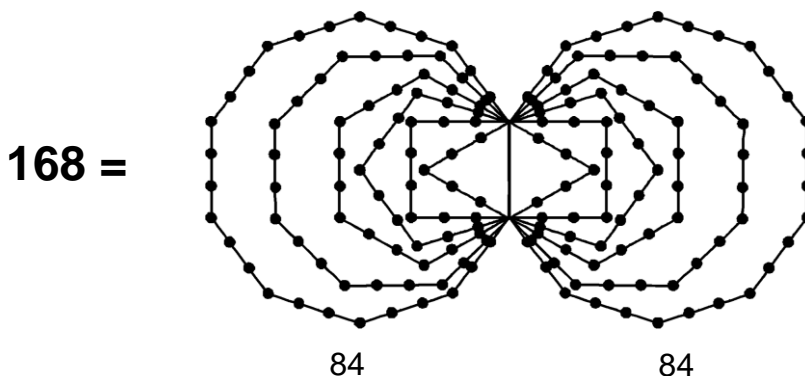


Figure 6. **168** yods form the boundaries of the first (6+6) polygons of the inner form of the Tree of Life.

(Fig. 5), the sum of the cubes of 1, 2, 3 & 4 at its four corners is 100 and the sum of the 16 other integers is 250. The tetrahedron embodies the interval composition of the seven modes.

5. Pattern of modal intervals in superstring oscillations

The seven modes have 130 rising, diatonic intervals other than 1, of which 46 comprise the minimum number of *different* diatonic intervals, these varying from mode to mode. There are $(130-46=84)$ such diatonic intervals that are repetitions of the basic set. Similarly, there are 84 falling, diatonic intervals with reciprocal tone ratios that are repetitions. There are $(84+84=168)$ repetitions of the diatonic intervals and their inverses. The number value **168** of *Cholem Yesodoth*, Mundane Chakra of Malkuth, is the number of diatonic intervals and their inverses that are repetitions of any of the seven basic diatonic intervals other than 1 and their inverses. This

84:84 division is embodied in the inner Tree of Life as the **168** yods lying on the sides of the first (6+6) enfolded polygons, 84 on either side of the root edge (Fig. 6).

The 2nd-order tetractys has 84 yods surrounding its centre (Fig. 7), where $84 = 1^2 + 3^2 + 5^2 + 7^2$.

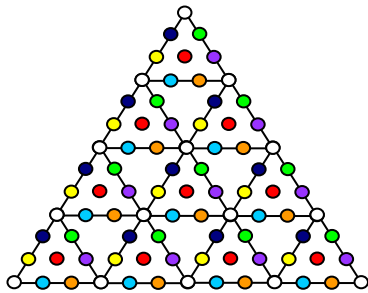


Figure 7. The 2nd-order tetractys.

This means that any aspect of physical reality that embodies cosmic wholeness symbolised by the tetractys is expressed by 84 degrees of freedom in this reality, which is symbolised by the central yod of the central 1st-order tetractys of the 2nd-order tetractys. This number accords with

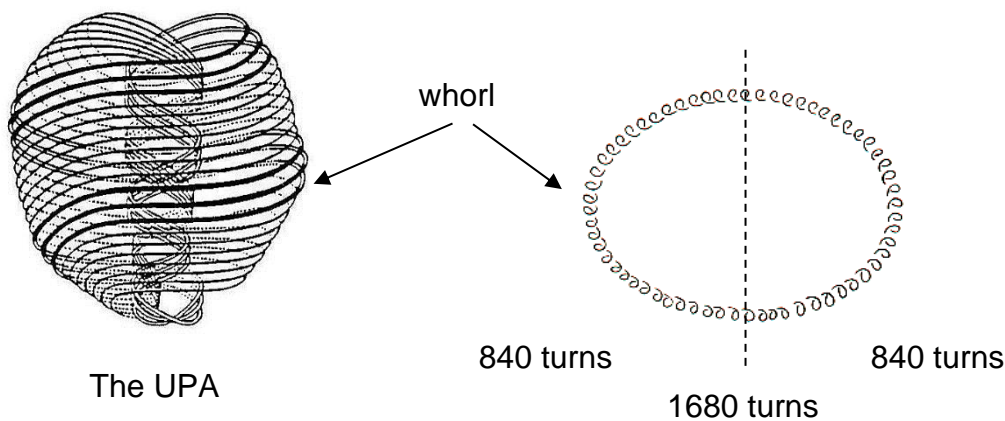


Figure 8. Each of the ten whorls of the UPA makes five revolutions around its axis as it coils 1680 times, that is, **168** times in each half-revolution and 84 times in the outer or inner half of the UPA.

the Tetrad Principle because 1, 3, 5 & 7 are the first four odd integers.

The 84:84 division corresponds in the subquark superstring to the inner and outer halves of the UPA, each of which comprises $2\frac{1}{2}$ revolutions of ten whorls with 840 1st-order spirillae in each one (Fig. 8). Every ten spirillae can be said to correspond to a diatonic interval between the notes of the seven Greek modes, the outer and inner halves corresponding to the intervals and their inverses. It is as if the circularly polarised oscillations of each whorl reproduce all the diatonic intervals between different notes of the seven musical modes.

6. Comparison of modal intervals with the octonions

This conclusion in the context of superstring structure returns us full circle to the discussion in Article 15 of the similarity between the 8-note scale and the eight unit octonions, as now explained. The octonions, or Cayley numbers, form the fourth class of division algebras. There are one real unit octonion, 1, and seven imaginary unit octonions e_i ($i = 1-7$) with the property $e_i^2 = -1$ ($i = 1-7$). Their multiplication is anti-commutative:

$$e_i e_j = -e_j e_i \quad (i \neq j)$$

Table 12 is the multiplication table for the 8-tuple of unit octonions: $(1, e_1, e_2, e_3, e_4, e_5, e_6, e_7)$.

Table 12. Table of multiplication of pairs of unit octonions

\cdot	e_1	e_2	e_3	e_4	e_5	e_6	e_7	1
e_1	-1	e_4	e_7	$-e_2$	e_6	$-e_5$	$-e_3$	e_1
e_2	$-e_4$	-1	e_5	e_1	$-e_3$	e_7	$-e_6$	e_2
e_3	$-e_7$	$-e_5$	-1	e_6	e_2	$-e_4$	e_1	e_3
e_4	e_2	$-e_1$	$-e_6$	-1	e_7	e_3	$-e_5$	e_4
e_5	$-e_6$	e_3	$-e_2$	$-e_7$	-1	e_1	e_4	e_5
e_6	e_5	$-e_7$	e_4	$-e_3$	$-e_1$	-1	e_2	e_6
e_7	e_3	e_6	$-e_1$	e_5	$-e_4$	$-e_2$	-1	e_7
1	e_1	e_2	e_3	e_4	e_5	e_6	e_7	1

There are 28 octonions in the violet triangle and 28 octonions in the blue triangle. The **21** octonions formed by multiplication of pairs of $e_1, e_2 \dots e_7$ in the dashed triangle on one side of the diagonal of the square array are the negatives (that is, reciprocals, because $e_i^2 = -1$, so that $-e_i = 1/e_i$) of their **21** counterparts in the dashed triangle on the other side of the diagonal. There are **21** octonions with positive signs and **21** with negative signs. The Godname *El Chai* with number value **49** prescribes the number of products of imaginary octonions, the Godname *Ehyeh* with number value **21** prescribing how many of them have positive signs and how many have negative signs, that is, how many are the reciprocals of them.

Compare this with the intervals between the notes of the Pythagorean scale shown in Table 13.

Table 13. Table of intervals of the Pythagorean musical scale

	C	D	E	F	G	A	B	C'
C	1	9/8	81/64	4/3	3/2	27/16	243/128	2
D	8/9	1	9/8	32/27	4/3	3/2	27/16	16/9
E	64/81	8/9	1	256/243	32/27	4/3	3/2	128/81
F	3/4	27/32	243/256	1	9/8	81/64	729/512	3/2
G	2/3	3/4	27/32	8/9	1	9/8	81/64	4/3
A	16/27	2/3	3/4	64/81	8/9	1	9/8	32/27
B	128/243	16/27	2/3	512/729	64/81	8/9	1	256/243
C'	1/2	9/16	81/128	2/3	3/4	27/32	243/256	1

The pitch decreases of the 28 falling intervals (D-C, etc) in the violet triangle are the reciprocals of the pitch increases of the 28 corresponding rising intervals (C-D, etc) in the blue triangle. The eight diagonal entries 1 denote the zero pitch difference between each note and itself.

In both tables, the diagonal set of entries is like a mirror that reflects each entry into its image. Each entry in a triangle in Table 12 corresponds to an interval, a negative octonion being the counterpart of a falling interval because it is the reciprocal of an octonion, just as the decrease in pitch of a falling interval is the reciprocal of the increase in pitch of the same rising interval.

Multiplication of two imaginary octonions: $e_i \times e_j$ ($j > i$) corresponds to the interval between two rising notes i and j . Multiplication in the reverse order corresponds to the interval between the notes j and i in order of descending pitch. Non-commutativity of octonion multiplication: $e_i e_j = -e_j e_i$ corresponds to the pitch decrease of a falling interval being the reciprocal of that of the same rising interval.

The seven imaginary octonions form seven 3-tuples:

1. (e_1, e_2, e_4)
2. (e_2, e_3, e_5)
3. (e_3, e_4, e_6)
4. (e_4, e_5, e_7)
5. (e_5, e_6, e_1)
6. (e_6, e_7, e_2)
7. (e_7, e_1, e_3)

It was proposed in Article 15¹³ that these 3-tuples are the octonion equivalent of the seven Greek octave species because they each form seven combinations of one, two and three octonions in analogy to the seven notes of each species. Furthermore, it was shown that the seven 3-tuples have 84 permutations of pairs and triplets and that the seven 3-tuples of negative octonions

14 slices
 12 coloured triangles per slice
 $14 \times 12 = 168$ coloured triangles
 in 24 heptagons

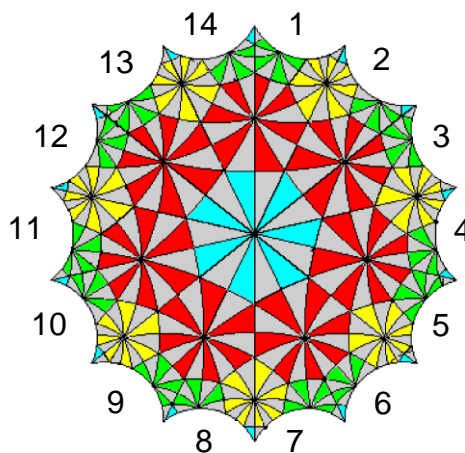


Figure 9. The Klein Configuration.

(inverses of octonions) have 84 permutations of their pairs and triplets, a total of **168** permutations. Each 3-tuple corresponds to an octave species. Each mode has on average 12 repetitions of its diatonic intervals (see below) and each of these intervals correspond to a permutation.

The same pattern is displayed in the Klein Configuration (Fig. 9) of the **168** elements of the group of transformations of the Klein Quartic Equation, which is well known to mathematicians.¹⁴ Each of the 14 slices shown numbered has 12 coloured triangles. The slices are of two types, each type containing 12 triangles arranged in the whole configuration as 24 heptagons. The seven slices of one type correspond to the repeated, rising intervals of the seven octave species and the seven slices of the other type correspond to their repeated, falling intervals, the 12 coloured triangles in a slice corresponding to the average number of 12 repetitions of diatonic intervals or their inverses.

Such precise parallelism between the seven musical modes and an object that is regarded in the

mathematical world as possessing great beauty is evidence of the archetypal Ideas of the Divine Mind being realised in two contexts that appear superficially to have nothing in common, namely, music and the M-theory underlying superstring physics, which physicists are currently seeking in order to unify their understanding of the forces of nature in a single theory. However, sound is a vibration of the air and, just as a plucked string creates musical notes of different frequencies, so superstrings vibrate in a variety of ways to create all the subatomic particles of which they are components. The form of a superstring is therefore generated by a process analogous to playing the intervals — both rising and falling — between (as shown below) the notes D, E, F, G & A of the seven musical modes.

By inspecting the table of intervals for each mode, counting its number of diatonic intervals and subtracting the number of different diatonic intervals indicated under each table, the numbers of repetitions of these intervals for individual modes are found to be:

Hypolydian	Mixolydian	Lydian	Dorian	Hypodorian	Phrygian	Hypophrygian
13	11	12	12	11	13	12

Total = 84.

Average = $84/7 = 12$.

In terms of notes, the composition of the 84 diatonic intervals is:

D	E	F	G	A
28	11	23	17	5

The interval 243/128 for note B and the interval 2 for note C' are absent because they are the only ones that never appear more than once in a mode (Phrygian, Lydian, Hypophrygian and Hypolydian) and so have no repetitions. Five different intervals comprise the set of 84. This 5:1 division has the following analogy with the Tree of Life: The lowest five Sephiroth of Construction:

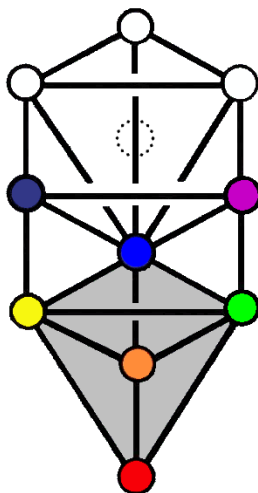


Figure 10. Only 5 of the 7 notes below the octave are repeated. This is analogous to how the five lowest Sephiroth of Construction of the Tree of Life form its Lower Face (shown shaded).

Malkuth, Yesod, Hod, Netzach & Tiphareth (Fig. 10) constitute what in Kabbalah is called its 'Lower Face,' or 'Microposopus.' They formally correspond to the five types of intervals that are repeated 84 times. The octave and note B, which are not repeated, formally correspond, respectively, to Chesed and Geburah, which are unshared with adjacent members of a set of overlapping Trees of Life, that is, they are *intrinsic* to their own trees.

This 5:2 division is reflected also in the five tone intervals (T) and two leimmas (L) of each mode:

T L T T T L T	Dorian
L T T T L T T	Phrygian
T T T L T T L	Lydian
T T L T T L T	Mixolydian
T L T T L T T	Hypodorian
L T T L T T T	Hypophrygian
T T L T T T L	Hypolydian

It is also manifested by the seven modes themselves because, regarding the 14 notes that they span as a cycle that starts again on the second octave, five successive cycles generates **15** harmonics and seven cycles creates **26** harmonics, where **15** is the number value of the Godname *Yah*, the older version of the Godname *Yahweh*, whose number value is **26**.

In traditions of esoteric knowledge, the lowest five Sephiroth of Construction have the following correspondence with the five Elements:

Tiphareth	Aether
Netzach	Fire
Hod	Air
Yesod	Water
Malkuth	Earth

The very diatonic intervals that correspond to the five Elements are therefore those whose repetitions and their inverses number **168**, the number value of *Cholem Yesodoth*, Mundane Chakra of the lowest Sephirah, Malkuth.

The **168** automorphisms of the Klein Quartic are represented by the **168** coloured, hyperbolic

$$\begin{aligned}
 168 &= 7 \times 24 = 7 \times [(1 + 2) + (4 + 3) + (4 + 3) + (4 + 3)] \\
 &= 7(4 + 4 + 4) + 7[(3 + 3 + 2) + (3 + 1)] \\
 &= (28 + 28 + 28) + (56 + 28) \\
 &= 84 + 84.
 \end{aligned}$$

triangles of the Klein Configuration:

Seven slices of one type have 84 triangles made up of 28 red, 28 yellow and 28 green triangles and the seven slices of the other type have 84 triangles made up of **21** yellow, **21** green, **21** red and **21** blue triangles. Compare the former with the fact that the 84 diatonic intervals between notes D, E, F, G & A comprise 28 seconds, 11 thirds, 23 perfect fourths, 17 perfect fifths and 5 sixths, that is, 28 seconds, 28 thirds and fifths and 28 perfect fourths and sixths. For the 84

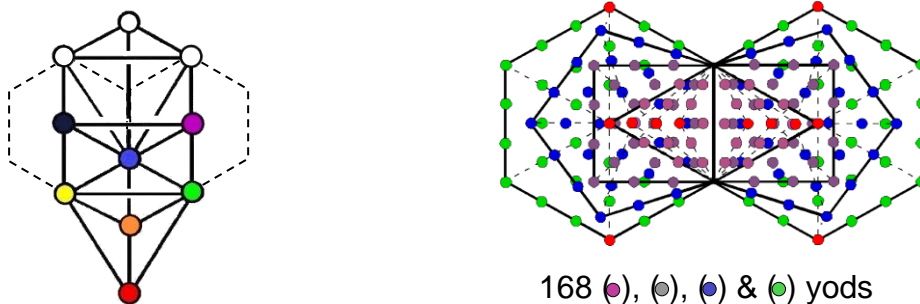


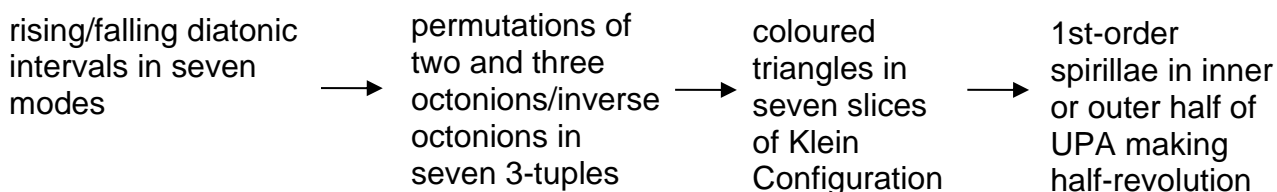
Figure 11

90 yods in four enfolded polygons outside the root edge.
 (6+6) (●) yods either centres of polygons or coinciding with the Sephiroth of the Tree of Life.
 84 other yods outside root edge in each set of four polygons.

coloured triangles belonging to seven slices of one type, the same pattern of three groups of 28 exists in both cases, although which sets of 28 coloured triangles correspond to which groups of 28 intervals cannot be ascertained at this stage. The four important features to notice are:

1. the Klein Configuration naturally divides the number **168** into two sets of 84 as its two types of slices have a total of 84 hyperbolic triangles;
2. the **168** diatonic intervals between notes D, E, F, G & A of the seven modes divide into 84 ascending and 84 descending intervals (the 84 entries on one side of the diagonal of their tables of intervals and 84 entries on the other side);
3. the seven 3-tuples of octonions have 84 permutations of two and three octonions and 84 permutations of two and three of their inverses;
4. the **168** 1st-order spirillae in each half-revolution of a whorl of the UPA comprise 84 in its outer half and 84 in its inner half.

They indicate the following correspondences:



The division of the Klein Configuration into two types of slices corresponds to the distinction between rising and falling intervals and to the difference between octonions and their inverses. The profound consequences of this analogy will be explored in a future article.

It was found earlier that the first four polygons of the inner form of the Tree of Life have 90 yods outside their root edge that correspond to the different intervals (including 1) of the seven Greek musical modes. Four of these yods are their centres and two (the top and bottom corner of the hexagon) coincide with Chokmah and Netzach (Fig. 11). There are (90–4–2=84) other yods in

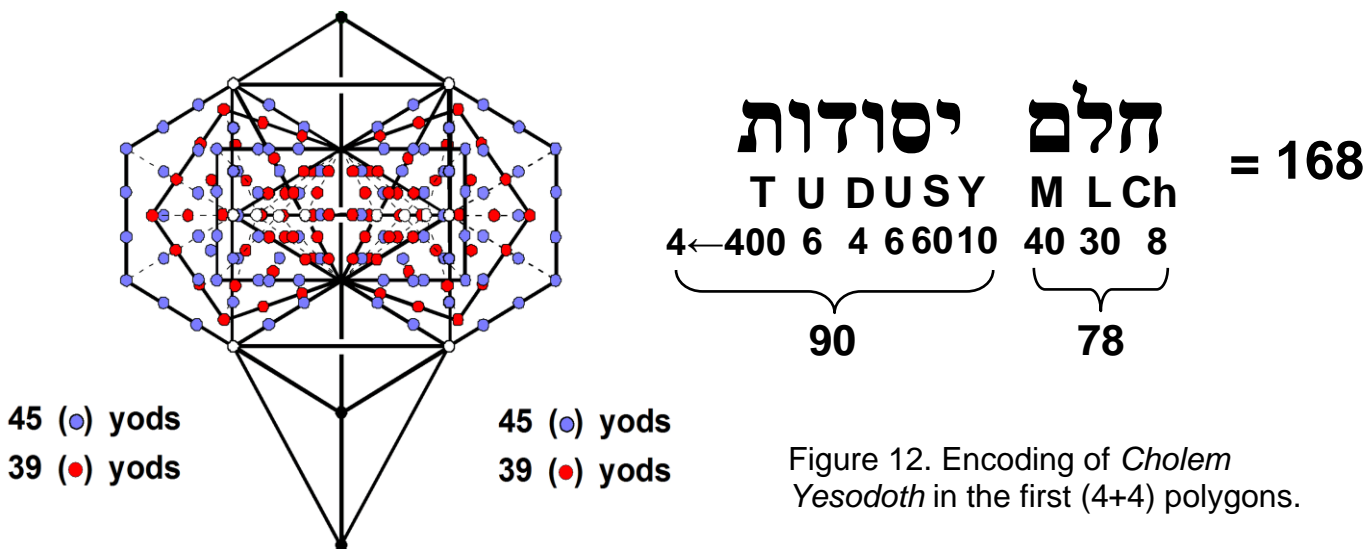


Figure 12. Encoding of *Cholem Yesodoth* in the first (4+4) polygons.

these polygons outside their root edge, both sets of four polygons having (84+84=168) such yods. The yods in one set of polygons correspond to the coloured triangles in the seven similar slices of the Klein Configuration, repeated, rising diatonic intervals between the notes D, E, F, G, & B in the seven modes, permutations of two and three octonions in the seven 3-tuples and 1st-order spirillae in one-half of a revolution of the inner or outer half of the UPA. The yods in the other, mirror image set of polygons correspond to the coloured triangles in the other set of similar slices

of the Klein Configuration, repeated, descending diatonic intervals, permutations of two and three inverse octonions in their seven 3-tuples and 1st-order spirillae in a half of a revolution of the outer or inner half of the UPA. The musical counterpart of the two identical halves of the inner form of the Tree of Life is the distinction between rising and falling intervals. Its octonion counterpart is the difference between the imaginary octonions and their inverses. Its counterpart in the Klein Configuration is the difference between its two types of slices (their mathematical difference is too technical to discuss here).

Remarkable confirmation that it is not coincidental that the first (4+4) polygons should have **168** yods other than their centres and those coinciding with the positions of Sephiroth comes from the way in which the number values of the Hebrew words 'Cholem' and 'Yesodoth' summing to **168** are themselves embodied in the yod populations of these polygons (Fig. 12). The numbers of yods in the triangle, square, pentagon and hexagon that are outside the root edge and which are not centres of polygons or coincident with the positions of Sephiroth of the Tree of Life when the latter is projected onto the plane containing the polygons are listed below:

triangle	13	⌋	
square	20	---39---	
pentagon	26	⌋	45
hexagon	25	-----'	
	<u>84</u>		
	Total =		<u>84</u>

The triangle and pentagon have 39 yods and the square and hexagon have 45 yods. Hence, both sets of triangles and pentagons have 78 yods and both sets of squares and hexagons have 90 yods. The number value of *Cholem* is 78 and the number value of *Yesodoth* is 90. The very numerical values of the Hebrew words of the Mundane Chakra of Malkuth are encoded geometrically in the first (4+4) polygons! This is an example of the Tetrad Principle,¹⁵ wherein the fourth member or the first four members of a sequence of mathematical objects always embody a number that has cosmic significance, such as the number of space-time dimensions, the number of particles transmitting superstring forces or the oscillatory form of a superstring.

Set out below are the ways in which properties of the set of four polygons and the (4+4) polygons are prescribed by the number values of the Godnames of the ten Sephiroth:

- Kether: **21** **21** geometrical elements in the (4+4) polygons are shared with the 1-tree;*
- Chokmah: **15** 4 enfolded polygons have **15** sides and **15** corners of their 17 tetractyses. The number of yods on the boundaries of the 34 tetractyses of the (4+4) polygons = 150 = **15**×10. The (4+4) polygons share **15** corners and sides with the Tree of Life;
- 26** (4+4) enfolded polygons have **26** corners outside their root edge. Every 4 enfolded polygons have **26** corners and sides. The (4+4) enfolded polygons share **26** corners and sides with the Tree of Life;
- Binah: **50** number of corners and sides of (4+4) enfolded polygons = 51 = **50**th integer after 1. (4+4) enfolded polygons have **50** sides unshared with the 1-tree;
- Chesed: **31** number of sides of 17 tetractyses in 4 enfolded polygons = **31**. Tree of Life has **31** geometrical elements unshared with (4+4) enfolded polygons;
- Geburah: **36** 4 separate polygons have **36** corners and sides and **36** sides of their 18 tetractyses. The polygons have **36** hexagonal yods on their boundaries. (4+4) separate polygons have **36** corners and **36** sides;
- Tiphareth: **76** 4 separate polygons have **76** geometrical elements. (4+4) enfolded polygons have **76** yods outside their root edge on their boundaries and 152 yods unshared with the 1-tree, where 152 = **76**th even integer;

* The 1-tree is the lowest tree in a set of overlapping Trees of Life.

Netzach: **129** (4+4) enfolded polygons have 130 yods on their boundaries other than corners outside their root edge, where $130 = 129\text{th}$ integer after 1;
 Hod: **153** (4+4) enfolded polygons have 154 hexagonal yods outside their root edge, where $154 = 153\text{rd}$ integer after 1;
 Yesod: **49** every (4+4) enfolded polygons have **49** corners and sides;
 Malkuth: **65** 66 hexagonal yods in the 4 enfolded polygons are unshared with the 1-tree, where $66 = 65\text{th}$ integer after 1;
155 (4+4) enfolded polygons have 156 yods outside their root edge unshared with the Tree of Life, where $156 = 155\text{th}$ integer after 1.

The Godnames define that section of the inner form of the Tree of Life blueprint that encodes the structural parameter **168** of superstrings. Actually, they prescribe other sections as well that embody the same number (e.g., the first (6+6) polygons).¹⁶ This is because of the fractal quality of the inner form of the Tree of Life, in which information about the physical manifestation of the blueprint is encoded in different but equivalent sections of it.

Conclusion

In coming to the consensus (or almost so) that the ancient Greek musical modes were likely to have been different keys of a single scale, scholars have either ignored or been oblivious of ancient writers who told of how Pythagoras, Plato and Greek musicians learned from the ancient Egyptians, who, according to Nichomachus, ascribed 28 sounds to the universe. This meant that the latter — and almost certainly also the former — knew about the seven octave species with 28 notes. This makes it highly likely, if not certain, that the Greek modes were different scales and that their patterns of intervals were known to those with musical learning, for why would ancient Greeks have played all their music in different keys of one scale if some of them knew of seven such scales? The fact that early Christian musical theorists appropriated Greek modal labels in their derivation of the church modes from the Byzantine *octoechoi* does not have the implication that the ancient Greeks did not understand their modes as scales. All it means is that the church modes were not directly inherited from Greek octave species. To conclude from this fact that the Greek modes were not scales is simply a non sequitur.

The ten Godnames prescribe different sets of intervals between the notes of the seven scales. The reason for this is that the seven octave species constitute a *whole* — both mathematically and spiritually. The tetractys version of Plato's Lambda has ten numbers that sum to 90. This is the number of different intervals between the notes of the seven scales. Just as four numbers — 1, 2, 4/3 & 3/2 — generate the Pythagorean scale and therefore all seven scales with a minimum number of 90 different intervals, so four yods lie on the shared base of the first four polygons of the inner form of the Tree of Life, having 90 yods outside it. The fact that the four inclined rows of the Lambda tetractys add up to the yod population of each polygon confirms that this remarkable correspondence is not coincidental. The tetrahedral generalisation of the Lambda tetractys has 20 numbers that sum to the number of intervals between notes above the tonic of each scale. There are 84 copies of the rising, diatonic intervals in the seven scales and similarly for the falling, diatonic intervals. This 84:84 pattern is present in the first (4+4) polygons as their (84+84) yods that are neither centres of polygons nor coincident with the positions of Sephiroth of the Tree of Life. Confirmation that this is not coincidental is the fact that the number values of the two words making up the Hebrew name of the Mundane Chakra of Malkuth with number value **168** are the yod populations of pairs of polygons in the set of four and their counterparts on the other side of their shared edge. The ten Godnames prescribe the (4+4) polygons as a Tree of Life pattern encoding the superstring structural parameter **168**.

The eight unit octonions are the counterpart in number theory of the eight notes of the musical scale. The real unit octonion corresponds to the octave and the seven imaginary unit octonions correspond to the seven notes below it. Multiplication of pairs of octonions is equivalent to the

rising interval between two notes. Division is equivalent to their falling interval, that is, to the interval between them in reverse order. Anticommutivity of their multiplication corresponds to the pitch difference of two notes being the reciprocal of that of the two notes in reverse order. The counterpart in the Klein Configuration of the **168** copies in the seven musical scales of the basic set of rising and falling diatonic intervals is its **168** triangles denoting the automorphisms of the Klein Quartic. Its seven slices of one type with 84 triangles correspond to the set of 84 rising, diatonic intervals in the seven scales and the seven slices of the other type with 84 triangles correspond to their set of 84 falling, diatonic intervals. The triangles denote the **168** elements of $PSL(2,7)$. As every element of a group has its inverse, the rising and falling diatonic intervals correspond to elements of $PSL(2,7)$ and their inverses. The 84 permutations of two and three octonions in the seven 3-tuples and the 84 similar permutations of their inverses are the octonion counterparts of the 84 copies of rising intervals and the 84 copies of falling intervals that are diatonic.

The seven notes above the tonic of the Pythagorean scale and the seven unit imaginary octonions in the fourth division algebra are analogous in ways beyond coincidence because they are examples of the seven-fold nature of the phenomenal aspect of God manifested in the world, as abstractly symbolised by the seven hexagonal yods of the tetractys and as expressed in the Tree of Life by its seven Sephiroth of Construction. Music is patterns of sound frequencies, or real numbers (the first division algebra), ordered in time, i.e., pairs of numbers (frequency, time coordinate) that are analogous to the complex numbers (the second division algebra). Sound is a vibration of a three-dimensional medium taking place in time, i.e., sets of four numbers define its displacement, analogous to the four-dimensional quaternions (the third division algebra). The eight-dimensional octonions associated with vibrating superstrings are the fourth and last division algebra. In conformity with the title “holding the key of nature” assigned by the Pythagoreans to the Tetrad, the vibrations of the macrocosm and the microcosm are connected by *four* classes of number.

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